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Kernels and Operators on the Space of Continuous Functions.

Francis Dennis Sentilles Jr

Louisiana State University and Agricultural & Mechanical College

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Kernels and Operators on the
Space of Continuous Functions

A Dissertation

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Francis Dennis Sentilles, Jr.
B.S., Francis T. Nicholls State College, 1963
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ABSTRACT

A kernel of measures is a mapping λ on a topological space T into the space of measures $M(S)$ on a locally compact space S , such that the function $\lambda(f)(x) = \int_S f(y) \lambda(x, dy)$ belongs to $C(T)$ when f belongs to $C_0(S)$. This notion has been studied extensively in probability theory with great restrictions on the spaces S and T and the range of λ . The primary concern of this dissertation is the removal of these restrictions, with applications to topics distinct from probability as a consequence.

The first chapter consists entirely of preliminary material. In Chapter II it is shown that $\lambda(x)(E) = \lambda(x, E)$ is a Borel measurable function of x in T for each Borel subset E of S . When T is also locally compact and Hausdorff it is then shown that the measure $\lambda(\mu)(E) = \int_T \lambda(x, E) \mu(dx)$ is a regular measure on S for μ in $M(T)$. These results underlie all succeeding work. As a consequence, a continuous operator A on $C_0(S)$ into $C(T)$ can be written as $Af = \lambda(f)$, for $f \in C_0(S)$, and for $\mu \in M(T)$, one has additionally that $A^*\mu = \lambda(\mu)$ where A^* is the adjoint map of A on $M(T)$.

The third chapter is devoted to the study of linear maps A on $C(S)$ into $C(T)$ and similar integral representations are obtained for A and A^* when A is continuous on $C(S)_\beta$. Necessary and sufficient conditions on the kernel λ are given so that the formula $Af = \lambda(f)$ defines a continuous operator on

$C(S)_\beta$ into $C(T)_\beta$. Extensions of an operator from $C_0(S)$ to $C(S)$ are also studied. It is shown that a norm continuous operator on $C(S)$ given by a kernel must be continuous on $C(S)_\beta$ when S is paracompact.

It is in Chapter IV that the removal of hypotheses on the spaces S and T yield results of wide application. A study is made of kernels λ for which $\lambda(\cdot, E)$ is continuous for Borel sets E . This is shown to be equivalent to continuity of $\lambda(\cdot, C)$ for closed sets C whose complement is σ -compact and equivalent to the weak continuity of the mapping $x \rightarrow \lambda(x)$ or to the weak compactness of the operator $Af = \lambda(f)$ and finally, equivalent to the weak compactness of the sets $\{\lambda(x) : x \in K\}$ for K a compact subset of T . This result yields generalizations and improvements of well-known theorems on weak compactness and convergence in $M(S)$ and in the L^1 spaces. Several distinct kernels are studied as exemplary. A brief study is made of the norm continuity of $x \rightarrow \lambda(x)$ with analagous results, and finally of the continuity of $|\lambda|(\cdot, E)$, the variation of $\lambda(\cdot, E)$.

The fifth chapter is devoted to a study of the compact and weakly compact operators on $C(S)_\beta$. It is shown that a linear operator A maps a β -neighborhood of zero into a weakly relatively compact subset of $C(T)_\beta$ if and only if A is continuous with the norm topology on $C(T)$ and maps bounded sets into weakly relatively compact sets. Analagous results hold for compact operators.

In Chapter VI it is shown that every semigroup of continuous operators on $C(S)_\beta$ is generated by a transition function on $M(S)$. The β -equicontinuous semigroups on $C(S)$ are characterized with an application to semigroups of maps on $C(S)$.

INTRODUCTION

The concept of a kernel of measures on a topological space S has a long and varied history in mathematics. Classically, one studied operators A acting on functions f by means of the integral transformation $[Af](x) = \int_S f(y)K(x,y)\mu(dy)$, the kernel being the measures $\lambda(x,E) = \int_E K(x,y)\mu(dy)$, so that $[Af](x) = \int_S f(y)\lambda(x,dy)$. It is in this latter form that kernels appear when applied to modern mathematics. This is in particular true in the study of Markov and stochastic processes.

Although kernels of this type have been extensively studied by probabilists and others interested in such matters, the idea of a kernel has hardly been studied at all with a view towards its application to modern analysis, save in the case of kernels of vector valued measures as found in [9] and [13]. In this dissertation we undertake such a study.

A severe limitation on the applicability of those results found in probability theory to modern analysis is that the underlying space S is assumed to be first countable and that the measures involved are usually probability measures. In Chapter II we discard these assumptions, restricting our work only to regular measures on locally compact spaces. We show that the function $\lambda(\cdot, E)$ is Borel measurable and that the measure $\nu(E) = \int_T \lambda(x, E)\mu(dx)$ is a regular Borel measure on S . This result underlies all succeeding work.

In the next chapter we apply these results to the representation and characterization of the continuous linear operators on $C(S)_\beta$ and to their adjoint mappings. In essence our work shows that the β -topology on $C(S)$ is of natural interest for the study of linear transformations of $C(S)$, and moreover, if S is paracompact and one is interested, as in probability theory, in the study of operators given by kernels, then the requirement that the operators be β -continuous warrants no additional assumptions on the operator or its kernel.

It is in Chapter IV that the relaxation of hypotheses on the space S and the measures involved yields wide application of the idea of a kernel. In particular, it is shown that the study of kernels λ for which $\lambda(\cdot, E)$ is continuous for Borel sets E , yields a general theory of weak compactness and convergence in the space of measures, and the L^1 spaces, and moreover, improves the known results. It is always the case in these applications that the underlying spaces are not generally first countable.

In Chapter V we make use of our work in the previous chapters in characterizing the compact and weakly compact operators on $C_0(S)$ and $C(S)_\beta$. The results obtained allow one to apply the general theory of such operators to the space $C(S)_\beta$, which is neither metrizable, bornological or barrelled unless S is compact.

Finally, in Chapter VI, we study semigroups of operators in the space $C(S)_\beta$. It is shown that a semigroup of continuous operators on $C(S)_\beta$ is generated by a transition function of regular measures and hence, under certain additional conditions, is generated by a Markov process on S . This is a result known for first countable spaces. We then characterize the equicontinuous semigroups on $C(S)_\beta$ so that one may apply the general theory to semigroups on this space. Our principle result is that when S is paracompact one may treat any semigroup on $C(S)$ generated by a Markov process, and satisfying the usual conditions, as an equicontinuous semigroup of continuous operators on $C(S)_\beta$ with no loss of generality and to some advantage.

In closing we mention that many of our proofs are considerably detailed; this was done solely to aid the reader and we trust that this detail will not become tedious. In Chapter I we include an introduction to our notation and terminology and a summary of those results upon which our work most depends. Finally, all theorems, definitions, and other such items are numbered consecutively with Theorem a.b denoting item number b in chapter a.

CHAPTER I

PRELIMINARIES

This dissertation is concerned with the study of only one topic, that of a kernel of measures, which we will presently define. The interest and application of this idea is found in several diverse fields however, and we attempt in this chapter to give the reader a brief foundation in each and to acquaint him with our notation and terminology.

Topology and function spaces.

Unless otherwise noted all topological notions are those of Kelley [20]. In particular the reader is urged to refer to this text with regard the idea of a net, a subnet and a cluster point of a net, which will be of some importance in the sequel. We assume a knowledge of locally compact Hausdorff topological spaces, paracompact and σ -compact spaces, Urysohn's lemma, and certain equivalences of compactness and net convergence. All of these may be found in detail in the above named source.

Let X be a locally compact Hausdorff space. We denote by $C(X)$ the collection of all bounded continuous complex-valued functions on X , by $C_0(X)$ those functions $f \in C(X)$ such that $\{x: |f(x)| \geq \epsilon\}$ is a compact subset of X for each $\epsilon > 0$, and by $C_c(X)$ those functions in $C_0(X)$ of compact support; that is, the closure of the set of non-zeroes of a function f in $C_c(X)$ is compact in X , and we denote this set by $\text{spt}(f)$.

If Δ is a collection of subsets of X we denote by $\sigma(\Delta)$ the σ -algebra generated by Δ . Thus $\sigma(\Delta)$ is the intersection of all sets \mathcal{E} of subsets of X such that $\Delta \subseteq \mathcal{E}$, $X \in \mathcal{E}$, $X \setminus A \in \mathcal{E}$ whenever $A \in \mathcal{E}$, and if $A_1, A_2, \dots \in \mathcal{E}$ then so are $\bigcap_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty} A_i$. If the set Δ consists of all open subsets of X , then the elements of $\sigma(\Delta)$ are called the Borel sets of X .

If a function f is real-valued on X we say that f is $\sigma(\Delta)$ -measurable if $\{x: f(x) > a\} \in \sigma(\Delta)$ for all real numbers a . A complex-valued function is measurable if its real and imaginary parts are. We denote by $B(X)$ the set of all bounded complex valued Borel measurable functions on X . Hence one has $C_c(X) \subset C_0(X) \subset C(X) \subset B(X)$.

If $f \in B(X)$ we set $\|f\| = \sup \{|f(x)| : x \in X\}$. In the topology induced by this norm, the spaces $C_0(X)$, $C(X)$, and $B(X)$ are complete normed vector spaces and $C_c(X)$ is dense in $C_0(X)$.

A real-valued function f will be called lower semicontinuous if $\{x: f(x) > a\}$ is an open set for all real numbers a , and upper semicontinuous if $\{x: f(x) < a\}$ is open. A function f is continuous if and only if it is both upper and lower semicontinuous. The sum of lower semicontinuous functions is again such, as is the supremum taken pointwise of any collection of lower semicontinuous functions. If f is lower semicontinuous, then $-f$ is upper semicontinuous. We refer the reader to Naimark [26] for the essential facts on such functions.

We end with a statement of Ascoli's theorem, a proof of which can be found in Edwards [13, p. 34]. Let us first define the compact-open topology on $C(X)$ as that topology on $C(X)$ with a base for the neighborhood system at a point $g \in C(X)$ consisting of all sets $U(g, K, \epsilon) = \{f \in C(X) : |f(x) - g(x)| < \epsilon \text{ for all } x \in K\}$ where $\epsilon > 0$ and K is a compact subset of X . A subset $F \subset C(X)$ is equicontinuous if given a point $x_0 \in X$ and $\epsilon > 0$ there is a neighborhood V of x_0 such that $|f(x) - f(x_0)| < \epsilon$ for all $x \in V$ and $f \in F$.

Theorem 1.1 (Ascoli). A set $F \subset C(X)$ is relatively compact in the compact-open topology on $C(X)$ if and only if F is equicontinuous and $\{f(x) : f \in F\}$ is a bounded subset in the complex plane for each $x \in X$.

Measure Theory.

Let Σ be a σ -algebra of subsets of X . A real-valued function μ defined on Σ is a real valued measure on Σ if $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for all collections $\{A_i\}$ of mutually disjoint sets in Σ . A measure on Σ is a function $\mu = \mu_1 + i\mu_2$ where μ_1 and μ_2 are real-valued measures on Σ .

If μ is a measure on Σ we set $|\mu|(E) = \sup \left\{ \sum_{i=1}^n |\mu(E_i)| : \{E_i\}_{i=1}^n \text{ is a partition by } \Sigma \text{ sets of } E \right\}$ for each $E \in \Sigma$. The number $|\mu|(E)$ is called the variation of μ on E and μ is called bounded if $|\mu|(X) < \infty$; in this case $|\mu|$ is a real valued measure on Σ by [9, p. 128].

If μ is a real valued measure on Σ then by [9, p. 130] there exist unique real measures, μ^+ and μ^- , taking on only

non-negative values, such that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$.

A bounded Borel measure μ is called regular if for each Borel set E and each $\epsilon > 0$ there is a compact set $K \subset E$ and an open set $U \subset E$ such that $|\mu|(U \setminus K) < \epsilon$. We denote by $M(X)$ the collection of all bounded regular Borel measures on X , and by $M(X)^+$ the positive measures therein. For $\mu \in M(X)$, the equality $\|\mu\| = |\mu|(X)$ defines a norm on $M(X)$ under which $M(X)$ is a complete normed vector space.

This last statement is proven in [9]. As a general reference for measure theoretic questions not specifically covered here we make use of [9] and also [17]. In particular, we assume a knowledge of the spaces $L^1(\mu)$ and $L_\infty(\mu)$ as found in [17], along with a familiarity with the Radon-Nikodym theorem as found in [17, p. 128].

If μ is a Σ -measure and f is a Σ -measurable function, we will use the notation $\int_X f d\mu = \int_X f(x) \mu(dx)$, where for $\mu \geq 0$ and $f \geq 0$, the integral is defined to be the supremum in \mathbb{R} of the numbers $\sum_{k=0}^{\infty} k/2^n \mu(E_k^n)$ where $E_k^n = \{x: k2^{-n} < f(x) \leq (k+1)2^{-n}\}$. The integral is extended to arbitrary measures and functions in the usual way.

If T is a topological space and S is a locally compact Hausdorff space, we will write $\lambda: T \rightarrow M(S)$ to mean that λ is a function on T whose range lies in $M(S)$. The value of $\lambda(x)$ at a Borel set E will be denoted by $\lambda(x, E)$ and the variation of $\lambda(x)$ on E by $|\lambda|(x, E)$. We also set $\|\lambda(x)\| = |\lambda|(x, S)$ and $\|\lambda\| = \sup\{\|\lambda(x)\|: x \in T\}$. Finally, if $f \in B(S)$ we set $\lambda(f)(x) =$

$\int_S f(y) \lambda(x, dy)$ for each $x \in T$. We will call the function λ a kernel provided that $\lambda(f) \in C(T)$ for all $f \in C_0(S)$. Our first task will be to study the functions $\lambda(\cdot, E)$ for E a Borel set.

Conway [6] has extended a result of Arens and Kelley[1] and shown that the extremal points of the unit sphere in $M(S)$ are the unimodular multiples of the unit point measures on S . We denote the point measure at $s \in S$ by δ_s^0 ; i.e., $\delta_s^0(E) = 1$ if $s \in E$, 0 if $s \notin E$. We will show that certain linear operators on $M(S)$ are completely determined by their images of the point measures. (See Corollary 2.22).

Before closing this section we mention that the idea of a kernel is not new and has been extensively studied in the theory of stochastic and Markov processes, though not in its full generality. Historically the study has been related to the representation of operators on the spaces of functions defined above. Because of the generalizations we will gain through our work it will be seen that a wider class of problems may be treated with this concept. In this respect the reader will find Chapter IV to be of interest.

Topological Vector Spaces and Functional Analysis.

In general our terminology will be that of [28], while our notation is principally that of [9]. We will make use of several results from Edwards [13], along with a good bit of motivation gained therein.

All topological vector spaces E will be locally convex and Hausdorff. That is, E is a vector space with a topology given by a collection $\{p\}$ of seminorms such that if $x \in E$ and $p(x) = 0$ for all p , then $x = 0$. A base for the neighborhood system at $0 \in E$ consists of all sets of the form $\{x: p(x) \leq 1\}$ where p is a seminorm on E . We denote by E^* the vector space of all continuous linear functionals on E . If $x \in E$ and $x^* \in E^*$ we denote by $\langle x, x^* \rangle$ the value of x^* at x .

The weak $*$ topology on E^* is denoted by $\sigma(E^*, E)$ and is defined by the family of seminorms $p_x(x^*) = |\langle x, x^* \rangle|$ for all $x \in E$. The weak topology on E , denoted by $\sigma(E, E^*)$, is defined by the seminorms $p_{x^*}(x) = |\langle x, x^* \rangle|$ for all $x^* \in E^*$.

A set $A \subset E$ will be called weakly relatively compact if its $\sigma(E, E^*)$ closure is compact in the topology $\sigma(E, E^*)$. A set A is bounded if for each seminorm p on E there is a number $\alpha > 0$ such that $p(x) \leq \alpha$ for all $x \in A$. The strong topology on E^* is the topology defined by the collection of seminorms $\{p_B\}$ defined by $p_B(x^*) = \sup\{|\langle x, x^* \rangle| : x \in B\}$, where B is a bounded set in E . If E is a normed space, then the seminorm p_B where $B = \{x: \|x\| \leq 1\}$ is a norm on E^* and E^* with the strong topology is a Banach space. Henceforth the symbol E^* will mean the dual space E^* of E with the strong topology.

Hence E^{**} denotes the adjoint of E^* with the strong topology on both spaces. We can embed E algebraically in E^{**} with the mapping $x \rightarrow x^{**}$ where $\langle x^*, x^{**} \rangle = \langle x, x^* \rangle$.

Finally, if $A \subset E$ we denote by A^0 the set of all $x^* \in E^*$ such that $|\langle x, x^* \rangle| \leq 1$ for all $x \in A$. A set $B \subset E^*$ is called equicontinuous provided $B \subset V^0$ for some neighborhood V of 0 in E .

For ease of reference we state without proof the following important results.

Theorem 1.2. (Uniform boundedness principle [9]). If E is a Banach space and $B \subset E^*$ such that $\sup\{|\langle x, x^* \rangle| : x^* \in B\} < \infty$ for each $x \in E$, then B is uniformly bounded. That is B is bounded in the strong topology on E^* .

Theorem 1.3. (Aliagou's theorem [28]). If V is a neighborhood of 0 in E , then V^0 is $\sigma(E^*, E)$ compact.

Theorem 1.4. (Eberlein's theorem [28]). Let E be a complete topological vector space. If every sequence of points of a subset A of E has a weak cluster point in E , then A is weakly relatively compact in E .

Let E and F be topological vector spaces and A a linear mapping of E into F .

Theorem 1.5. ([28, p. 38]). If A is continuous in the topologies $\sigma(E, E^*)$ and $\sigma(F, F^*)$, then the equation $\langle x, A^*x^* \rangle = \langle Ax, x^* \rangle$ defines a linear operator A^* from F^* into E^* which is continuous when E and F are normed spaces and conversely.

We will call the operator A^* the adjoint or transpose of A . We will make much use of all of the following results, many of which are easy to prove.

Theorem 1.6. ([28,p. 39]). If A is a continuous operator on E into F , then A is weakly continuous. Furthermore, if A is weakly continuous, then A^* is weakly continuous.

Theorem 1.7. ([28,p. 39]). If A is weakly continuous, then for $M \subset E$, $A^{*-1}(M^0) = A(M)^0$.

Theorem 1.8. ([13,p. 624]). Suppose the space F is complete and A is continuous. The following are equivalent.

- (a) A maps bounded sets in E into $\sigma(F, F^*)$ relatively compact sets in F .
- (b) A^* maps equicontinuous subsets of F^* into $\sigma(E^*, E^{**})$ relatively compact subsets of E^* .
- (c) $A^{**}(E^{**}) \subset F$.

Theorem 1.9. ([28,p. 152]). Let A be continuous on E into the complete space F . Then A maps bounded subsets of E into relatively compact subsets of F if and only if A^* maps equicontinuous subsets of F^* into relatively compact subsets of E^* .

Theorem 1.10. ([28,pp.67 and 34]). If A is a subset of E , then A is bounded if and only if A is weakly bounded. If A is convex, then the closure of A and its weak closure coincide.

This completes our summary of the general theory and we now turn to some specific spaces and their adjoints.

Throughout this paper the letter S denotes a locally compact Hausdorff space and $C_0(S)$ has the norm topology.

Theorem 1.11. (Riesz representation theorem [17]). If F is a linear functional on $C_0(S)$, then F is continuous if and only if there is a measure $\mu \in M(S)$ such that $F(f) = \int_S f d\mu$ for all $f \in C_0(S)$. Furthermore, μ is unique and $\|\mu\| = \|F\| = \sup\{|\int_S f d\mu| : \|f\| \leq 1\}$. If $F(f) \geq 0$ for all $f \in C_0(S)$, $f \geq 0$, then $\mu \in M(S)^+$.

Corollary 1.12. If U is an open subset of S and $\mu \in M(S)$ then $|\mu|(U) = \sup\{|\int_S f d\mu| : \|f\| \leq 1, f \in C_0(S) \text{ and } \text{spt}(f) \subset U\}$.

Consequently $C_0(S)^* = M(S)$. We also will consider $B(S)$ as a subset of $M(S)^*$ in the following sense. If $f \in B(S)$ then we identify f with the functional F on $M(S)$ defined by $F(\mu) = \int_S f d\mu$ for all $\mu \in M(S)$. As is easily seen this correspondence is linear and isometric in the respective norm topologies.

We wish to consider several topologies on the space $C(S)$. When we consider $C(S)$ with the norm topology we will always write $(C(S), \|\cdot\|)$. The β or strict topology on $C(S)$ is that locally convex topology on $C(S)$ defined by the seminorms $P_\phi(f) = \|\phi f\|$ for $\phi \in C_0(S)$; when we consider $C(S)$ with this topology we will write $C(S)_\beta$. It is easy to see that a base for the β -neighborhoods of 0 consists of all sets $V_\phi = \{f : P_\phi(f) \leq 1\}$ for $\phi \in C_0(S)$, $\phi \geq 0$. Consequently a β -equicontinuous subset of $C(S)_\beta^*$ is necessarily contained in some V_ϕ^0 for $\phi \geq 0$ in $C_0(S)$.

The β topology on $C(S)$ was introduced by Buck [2]. It has been studied by Glicksburg [15], and Wells [31], while Shields and Rubel [30] have found it to be of use in studying

the bounded analytic functions on the open unit disc. More recently, Conway [6] made a study of the relationship between $C(S)_\beta$ and its adjoint; the author recommends his work as an excellent summary of the results which we will most use here as well as an invaluable aid in the sequel.

We consider one further topology on $C(S)$, the β' or bounded strict topology introduced recently by Dorroh [8]. Let $B_r = \{f \in C(S) : \|f\| \leq r\}$. A base for the neighborhood system at 0 in the β' topology consists of all absolutely convex absorbent sets W such that for each $r > 0$ there is a β -neighborhood V of 0 such that $V \cap B_r \subset W$. For some general results on topologies generated in this manner see Collins [3]. Collins and Dorroh [4] have studied and made some use of the β' topology on $C(S)$ in their recent paper; we only make use of β' to yield results in $C(S)_\beta$.

We now summarize those properties of $C(S)_\beta$ and $C(S)_{\beta'}$ which will be of most use in the sequel, and also list some important results on weak compactness in the space of measures $M(S)$.

Theorem 1.13. (Buck[2]). A linear functional F on $C(S)_\beta$ is continuous if and only if there is a unique measure $\mu \in M(S)$ such that $F(f) = \int_S f d\mu$ for all $f \in C(S)$.

Theorem 1.14. (Buck [2]).

- (a) $C(S)_\beta$ is complete.
- (b) A set in $C(S)$ is β -bounded if and only if it is norm bounded.

- (c) The β and norm topologies agree if and only if S is compact.
- (d) On bounded subsets the β and compact-open topologies coincide.
- (e) $C_0(S)$ is β -dense in $C(S)$.
- (f) $C(S)_\beta^* = M(S)$.

The statement (f) of Theorem 1.14 of course follows from Theorem 1.13 and (b) since the norm topology of $M(S)$ is the topology defined by the norm bounded subsets of $C(S)$.

From our previous remarks it now follows that a set $H \subset M(S)$ is β -equicontinuous if and only if there is a $\phi \in C_0(S)$ with $\phi \geq 0$ such that $|\langle f, \mu \rangle| = |\int_S f d\mu| \leq \epsilon$ for all $f \in V_\phi$ and $\mu \in H$. A most important characterization of the β -equicontinuous sets is given by

Theorem 1.15. (Conway [5]). If $H \subset M(S)$ the following are equivalent.

- (a) H is β -equicontinuous.
- (b) H is uniformly bounded and for each $\epsilon > 0$ there is a compact set K of S such that $|\mu|(S \setminus K) < \epsilon$ for all $\mu \in H$.
- (c) For some $\phi \in C_0(S)$ with $\phi \geq 0$, $H \subset \{\mu \in M(S) : \mu \text{ vanishes off the non-zeroes of } \phi \text{ and } \|\frac{1}{\phi} \cdot \mu\| \leq 1\}$ where $(\frac{1}{\phi} \cdot \mu)(E) = \int_E 1/\phi d\mu$.

Theorem 1.16. (Conway [5]). If S is paracompact then every β -weak $*$ compact set (i.e., $\sigma(M(S), C(S)_\beta)$ compact set) is β -equicontinuous.

It has also been shown by Conway [6] that a β -weak $*$ compact set of non-negative measures is always β -equicontinuous for any space S .

All that we will need to know about the space $C(S)_{\beta'}$ is contained in the following.

Theorem 1.17. (Dorroh [8]).

- (a) $C(S)_{\beta'}$ is sequentially complete.
- (b) β' is a finer topology than β .
- (c) β and β' agree on norm bounded sets.
- (d) $C(S)_{\beta'}^* = M(S)$.
- (e) $\beta = \beta'$ if S is paracompact.

The last result follows from Conway [5] who has shown that when S is paracompact then β is the Mackey topology on $C(S)$; i.e., β is the finest locally convex topology on $C(S)$ for which the adjoint space is $M(S)$.

We now wish to state a few results on weak compactness and weak convergence in $M(S)$. The first result is due to Grothendieck [16] and can also be found in Edwards [13]. We will make much use of this result.

Theorem 1.18. (Grothendieck [16]). Let A be a bounded subset of $M(S)$. The following are equivalent.

- (a) If $f_n \rightarrow 0$ weakly in $C_0(S)$, then $\int_S f_n d\mu \rightarrow 0$ uniformly for $\mu \in A$.
- (b) If $\{U_n\}$ is a sequence of disjoint open subsets of S , then $\mu(U_n) \rightarrow 0$ uniformly for $\mu \in A$.

- (c) (1) For each compact set $K \subset S$ and each $\epsilon > 0$ there is an open set $U \supset K$ such that $|\mu|(U \setminus K) < \epsilon$ for all $\mu \in A$.
- (2) For each $\epsilon > 0$ there is a compact set $K \subset S$ such that $|\mu|(S \setminus K) < \epsilon$ for all $\mu \in A$.
- (d) A is weakly relatively compact.

We will make only slight use of the following theorem from Edwards [13, p. 287].

Theorem 1.19. If A is a weakly relatively compact set in $M(S)$ then there is a $\mu \in M(S)^+$ such that each $\nu \in A$ can be written in the form $\nu(E) = \int_E f d\mu$ for all Borel sets E , where $f \in L^1(\mu)$.

Definition 1.20. A set $A \subset M(S)$ is called uniformly outer (inner) regular with respect to (w.r.t) compact (open) sets if and only if for each $\epsilon > 0$ and each compact set K (open set V) there is an open set $U \supset K$ (a compact set $Q \subset V$) such that $|\mu|(U \setminus K) < \epsilon$ ($|\mu|(V \setminus Q) < \epsilon$) for all $\mu \in A$.

We provide a proof of the following result which is used in the proof of Theorem 1.18 and which we will have much need for in the proofs and statements in the sequel.

Theorem 1.21. A set $A \subset M(S)$ is uniformly inner regular w.r.t. open sets if and only if A is uniformly outer regular w.r.t. compact sets and uniformly inner regular for the set S .

Proof: We prove the necessity first. If $\epsilon > 0$ then since S is open there is a compact set $K \subset S$ such that $|\mu|(S \setminus K) < \epsilon$ for all $\mu \in A$. Suppose now that Q is a given compact set in S . Since $S \setminus Q$ is open there is a compact set $P \subset S \setminus Q$ such that

$|\mu|((S \setminus Q) \setminus P) < \epsilon$ for all $\mu \in A$ by hypothesis. Hence $|\mu|((S \setminus P) \setminus Q) < \epsilon$ for all $\mu \in A$ and $S \setminus P$ is an open set containing Q .

We now prove the sufficiency. Let $\epsilon > 0$, U an open set in S . By hypothesis there is a compact set Q such that $|\mu|(S \setminus Q) < \epsilon/2$ for all $\mu \in A$. Since $Q \setminus U$ is compact there is an open set $V \subset Q \setminus U$ such that $|\mu|(V \setminus (Q \setminus U)) < \epsilon/2$ for all $\mu \in A$. Let $Q_1 = Q \setminus V$. Then Q_1 is compact and $U \setminus Q_1 \subset U \setminus Q \subset V \setminus (Q \setminus U)$. Since $|\mu|(U \setminus Q) \leq |\mu|(S \setminus Q) < \epsilon/2$ this means $|\mu|(U \setminus Q_1) < \epsilon$ for all $\mu \in A$.

Theorem 1.22. (Conway [6]). If $\mu_n, \mu \in M(S)$, then the following are equivalent:

- (a) $\mu_n \rightarrow \mu$ weakly.
- (b) $\{\mu_n\}$ is uniformly bounded and $\mu_n(U) \rightarrow \mu(U)$ for all open sets U .
- (c) $\int f d\mu_n \rightarrow \int f d\mu$ for all bounded l.s.c. functions f .
- (d) (1) $\{\mu_n\}$ is β -equicontinuous and uniformly outer regular w.r.t. compacta.
 (2) $\mu_n \rightarrow \mu$ β -weak*; i.e. $\int_S f d\mu_n \rightarrow \int_S f d\mu$ for all $f \in C(S)_\beta$.

We will obtain a generalization of and improvement upon Theorem 1.22 in Chapter IV (see Theorem 4.10). We will also see (Corollary 3.3) that (2) of (d) can be weakened to " $\mu_n \rightarrow \mu$ weak* ". Conway [6] points out that (2) implies that $\{\mu_n\}$ is β -equicontinuous when S is paracompact.

Finally, suppose that for each $t \in [0, \infty)$ a kernel $\lambda_t: S \rightarrow M(S)$ is given, and that $\lambda_t(\cdot, E)$ is Borel measurable for all Borel

sets E and $\|\lambda_t\| < \infty$. We call the function $t \rightarrow \lambda_t$ a transition function if it satisfies the Chapman-Kolmogorov equation.

(1) $\lambda_{t+u}(x, E) = \int_S \lambda_t(y, E) \lambda_u(x, dy)$ for all $t, u \geq 0$, $x \in S$, and Borel sets E .

The transition function is a concept associated with Markov processes. Usually one takes λ_t to have range $M(S)^+$ and $\|\lambda_t\| \leq 1$ and $\lambda_0(x) \leq \overset{0}{x}$ for all $x \in S$. Then the measures $\lambda_t(x, \cdot)$ are considered as probability measures on the state space S . Kolmogorov [21] introduced this idea in 1931 and was motivated by the study of Brownian motion. One usually takes the number $\lambda_t(x, E)$ to be the probability that a moving particle in the space S which is at x at time zero is in E at time t . Perhaps the most well known transition function is that function which is associated with uniform motion along a straight line and is given by $\lambda_t(x, E) = (\overset{0}{x} + t)(E)$ where $S = [0, \infty)$. The reader may refer to the texts by Neveu [27], Loève [25] and Dynkin [11] for further information and a complete introduction to this concept. The second text by Dynkin [12] also contains a historical summary of the theory.

Semigroups of Operators.

Let X be a locally convex, sequentially complete, Hausdorff topological vector space and for each $t \in [0, \infty)$ let T_t be a continuous linear operator on X into itself. The collection $\{T_t : t \geq 0\}$ is called a semigroup of operators in X provided $T_{t+u} = T_t T_u$ for all $t, u \geq 0$.

The study of semigroups on Banach spaces originated with Hille [19] and Yosida [32] and is introduced and developed from various viewpoints in the texts by Dynkin [11], Dunford and Schwartz [9] and Hille and Phillips [19]. Dynkin also develops a theory of semigroups in the weak * topology on a Banach space, while Yosida [33] has since recorded the general theory of semigroups in such spaces X defined above; this theory was suggested by L. Schwartz [33,p. 249].

We present here certain basic results all of which are easily proven by a slight generalization of the proofs found in Dynkin for Banach spaces. With these results one can apply the results in [33] to the slightly more general case we consider here.

We will assume that the semigroup $\{T_t\}$ is equicontinuous on bounded intervals of $[0, \infty)$ throughout. That is, there is a number $a > 0$ such that for a given neighborhood V of 0 in X , there is a neighborhood U of 0 in X , such that $T_t(U) \subset V$ for all $t \leq a$. It follows that if for some $a > 0$ $\{T_t: t \leq a\}$ is equicontinuous, then $\{T_t: t \leq a\}$ is equicontinuous for all $a > 0$. This is a simple consequence of the semigroup property.

We let $X_0 = \{x \in X: \lim_{t \rightarrow 0^+} T_t x = x\}$ and let $Ax = \lim_{h \rightarrow 0^+} (T_h x - x)/h$ if this limit exists and set $D_A = \{x \in X: Ax \text{ is defined}\}$. The mapping $x \rightarrow Ax$ defines a linear operator A on the subspace D_A which is called the infinitesimal generator of the semigroup. The semigroup is said to be strongly continuous on X_0 .

Theorem 1.23. (1) X_0 is a closed subspace of X which is invariant under all operators T_t .

(2) If $x \in X_0$ then $t \rightarrow T_t x$ is a continuous function on $[0, \infty)$ into X_0 .

(3) $D_A \subset X_0$ and if $x \in D_A$ then $Ax \in X_0$.

(4) The closure of D_A is X_0 .

(5) If $x \in D_A$ then $T_t x \in D_A$ for $t \geq 0$ and $T_t x - x = \int_0^t T_s Ax \, ds = \int_0^t A T_s x \, ds$.

(6) If $x \in D_A$ then for all $t > 0$, $d/dt T_t x$ exists and equals $A T_t x$ and $T_t Ax$.

(7) The linear operator A restricted to X_0 is sequentially closed; i.e., if $\{x_n\} \subset D_A$ and $x, y \in X_0$ and $x_n \rightarrow x$ and $Ax_n \rightarrow y$ then $x \in D_A$ and $Ax = y$.

Since the space X_0 is closed and hence sequentially complete and is also invariant under all operators T_t , the restriction of the semigroup to X_0 yields a semigroup of operators to which the general theory of Yosida [33] nearly applies. The only additional requirement (see [33, p.234]) is that $\{T_t : t \geq 0\}$ be equicontinuous on X_0 . We will investigate this problem in detail for the specific spaces $C(S)_\beta$ or $C(S)_{\beta, \cdot}$.

Markov Processes.

Since a part of our results yield a connection between semigroups of operators and transition functions (see Chapter VI) and hence Markov processes, we give here a definition of this concept taken from Dynkin [11].

Let S be a set with a σ -algebra N . Let Ω be a set and $\xi: \Omega \rightarrow [0, \infty)$ and let $\Omega_t = \{\omega \in \Omega: \xi(\omega) > t\}$ and let M_t be a σ -algebra on Ω_t for each $t \geq 0$. For each $x \in S$, let P_x be a probability measure on a σ -algebra M^0 on Ω containing M_t for all $t \geq 0$. Finally let x_t be a function defined on Ω_t with range in S ; notice that $x(t, \omega) = x_t(\omega)$ is then a function of t defined on $[0, \xi(\omega))$.

The collection $\{\xi, P_x, M_t, x_t\}$ is said to form a Markov process provided the following conditions are satisfied.

- (1) If $t \leq u$ and $A \in M_t$, then $A \cap \Omega_u \in M_u$.
- (2) For $t \geq 0$, $x_t^{-1}(E) \in M_t$ for $E \in N$.
- (3) $P(t, x, E) = P_x(x_t^{-1}(E))$ is N -measurable in x for a fixed $t \geq 0$ and $E \in N$.
- (4) $P(0, x, E \setminus \{x\}) = 0$.
- (5) For $t, h \geq 0$, $E \in N$ and $A \in M_t$

$$P_x\{A \cap x_{t+h}^{-1}(E)\} = \int_{\Omega} \chi_A(\omega) P(h, x_t(\omega), E) P_x(d\omega).$$
- (6) For $\omega \in \Omega_t$ there is an $\omega' \in \Omega_0$ such that $x_u(\omega') = x_{t+u}(\omega)$ for $0 \leq u < \xi(\omega') = \xi(\omega) - t$.

The set Ω is called the sample space and the function ξ is called the lifetime of the process. The algebra M_t can be considered as the collection of events observed up to time t , where the functions x_t are visualized as the trajectories of moving particles in the space S . Finally $P_x(A)$ is the probability that a particle starting at the point $x \in S$ is in the set A .

It follows from (3), (4) and (5) that the formula $\lambda_t(x, E) = P(t, x, E)$ defines a transition function on S . Furthermore one can also prove that if f is a bounded N -measurable function on S then

$$\lambda_t(f)(x) = \int_S f(y) \lambda_t(x, dy) = \int_{\Omega} f(x_t(w)) P_x(dw).$$

Thus $\lambda_t(f)(x)$ represents what is usually called the expectation of the random variable (i.e., measurable function) $f \circ x_t$ with respect to the probability P_x .

CHAPTER II

KERNELS AND OPERATORS ON $C_0(S)$

Let T be a topological space, S a locally compact Hausdorff space, and let $\lambda: T \rightarrow M(S)$ be a kernel. We have seen in the introduction to transition functions that one wishes to form the $\int_S \lambda_t(y, E) \lambda_u(x, dy)$, where λ_t and λ_u are kernels. Consequently, one is interested in the integrability of the function $\lambda(\cdot, E)$ for Borel sets E . We determine conditions for this in terms of the functions $\lambda(f)$ for $f \in C_0(S)$. This serves as partial motivation for the work in this chapter.

Of further interest however is the natural manner in which kernels arise in the integral representation of operators. Let A be a continuous operator on $C_0(S)$ into $C(T)$ with the topology of pointwise convergence on T . For each $x \in T$, the mapping $f \rightarrow [Af](x)$ determines a bounded linear functional on $C_0(S)$, which, by Theorem 1.11, can be written in the form $[Af](x) = \int_S f(y) \lambda(x, dy)$ where $\lambda(x, \cdot) \in M(S)$. The resulting function $\lambda: T \rightarrow M(S)$ is a kernel. We will show that $(A^* \mu)(E) = \int_T \lambda(x, E) \mu(dx)$ and also that $A^{**}f = \lambda(f)$ for $f \in B(S)$. These results are crucial to the remainder of our work. We will, for example, use these results to determine necessary and sufficient conditions under which the operator A on $C_0(S)$ may be extended to an operator \bar{A} on $C(S)$ into $C(T)$ in a unique manner and such that \bar{A} is β -continuous.

We first consider the problem of Borel measurability of the function $\lambda(\cdot, E)$ for all Borel sets $E \subset S$.

We begin with a result due to Dynkin [10] which is crucial to our work in this section.

Theorem 2.1: Let Δ be a collection of subsets of a set X which is closed under finite intersections and let $X \in \Delta$. Let $\Sigma \subset \sigma(\Delta)$ and suppose Σ satisfies the conditions:

- (1) $\Delta \subset \Sigma$.
- (2) If $A, B \in \Sigma$ and $A \subset B$ then $A \setminus B \in \Sigma$.
- (3) If $\{A_i : i=1, 2, \dots\} \subset \Sigma$ and for $i \neq j$ we have $A_i \cap A_j = \emptyset$ then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.

Then $\Sigma = \sigma(\Delta)$.

Proof: Let us suppose that Σ satisfies the condition:

- (4) If $A, B \in \Sigma$ then $A \cap B \in \Sigma$.

Let $\{A_i\} \subset \Sigma$. Then it follows that $A_n \cap A_i \in \Sigma$ for each $i \leq n$ and by (2) $A_n \setminus (A_n \cap A_i) \in \Sigma$ and by induction and (4) so is

$$\bigcap_{i=1}^{n-1} A_n \setminus (A_n \cap A_i) = A_n \setminus \bigcup_{i=1}^{n-1} A_i \in \Sigma.$$

Let $E_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$. Then $\{E_i : i=1, 2, \dots\} \subset \Sigma$ and by (3)

$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i \in \Sigma$ since for $i \neq j$ we have $E_i \cap E_j = \emptyset$. Thus Σ is closed under arbitrary countable unions provided (4) holds.

Because of (1) and (2) it follows that Σ is closed under countable intersections and hence Σ is a σ -algebra which contains Δ and therefore we must have $\Sigma = \sigma(\Delta)$.

Thus the proof will be complete provided we can show (4) is valid. Let us say that a system Σ' satisfying the hypotheses (1), (2) and (3) and $\Delta \subset \Sigma' \subset \sigma(\Delta)$ satisfies condition P.

Let $\Sigma_1 = \bigcap \{\Sigma' : \Sigma' \text{ satisfies P}\}$. Then clearly $\Sigma \subset \Sigma_1$ and if we can show Σ_1 satisfies P and Σ_1 satisfies (4) we are through.

But clearly Σ_1 satisfies P. Let $\Sigma_2 = \{A \in \sigma(\Delta) : A \cap B \in \Sigma_1 \text{ for all } B \in \Delta\}$. We show Σ_2 satisfies P.

Since Δ is closed under finite intersections we have $\Sigma_2 \supset \Delta$ because $\Sigma_1 \supset \Delta$. If $A, B \in \Sigma_2$ and $A \supset B$ and $C \in \Delta$ then $A \cap C, B \cap C \in \Sigma_1$ and have by (2), $A \cap C \setminus B \cap C = A \setminus B \cap C \in \Sigma$. Hence $A \setminus B \in \Sigma_2$. If $\{A_i\} \subset \Sigma_2$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $B \in \Delta$ then $\{A_i \cap B\} \subset \Sigma_1$ and by (3) $\bigcup_{i=1}^{\infty} (A_i \cap B) = (\bigcup_{i=1}^{\infty} A_i) \cap B \in \Sigma_1$. Hence $\bigcup_{i=1}^{\infty} A_i \in \Sigma_2$. Thus Σ_2 satisfies P.

Hence $\Sigma_2 \supset \Sigma_1$. Now let $\Sigma_3 = \{A \in \sigma(\Delta) : A \cap B \in \Sigma_1 \text{ for all } B \in \Sigma_1\}$. If $A \in \Delta$ and $B \in \Sigma_1$ then $B \in \Sigma_2$ and hence by definition of Σ_2 , $A \cap B \in \Sigma_1$ and therefore $A \in \Sigma_3$. Thus $\Delta \subset \Sigma_3$ and as before Σ_3 must satisfy P and hence $\Sigma_3 \supset \Sigma_1$.

Thus let $A, B \in \Sigma_1$. Then since $A \in \Sigma_3$ then by definition, $A \cap B \in \Sigma_1$. Hence Σ_1 satisfies P and (4) and therefore $\Sigma_1 = \sigma(\Delta)$ and the proof is complete.

Corollary 2.2: If Σ is a collection of Borel sets which contains all the open subsets of a topological space X and satisfies (2) and (3) of Theorem 2.1 then Σ is the collection of all Borel sets.

We now begin a study of the functions $\lambda(\cdot, E)$ for E a Borel set. Let Δ denote the collection of open subsets of S .

Definition 2.3: Let $\lambda: T \rightarrow M(S)$. We call λ a quasi-kernel if $\lambda(x) \in M(S)^+$ for each $x \in T$ and if $\lambda(f)$ is a bounded lower semicontinuous function on T for each $f \geq 0$ in $C_0(S)$.

Lemma 2.4: If λ is either a kernel or quasi-kernel then $\|\lambda\| = \sup\{\|\lambda(x)\| : x \in T\} < \infty$ and $\|\lambda(f)\| \leq \|\lambda\| \|f\|$.

Proof: For each $x \in T$ let $F_x(f) = \lambda(f)(x)$. By hypothesis, for each $f \in C_0(S)$ $\sup\{|\lambda(f)(x)| : x \in T\} = \sup\{|F_x(f)| : x \in T\} = \|\lambda(f)\| < \infty$. Hence the collection of functionals $\{F_x : x \in T\}$ is pointwise bounded and hence uniformly bounded since $C_0(S)$ is a Banach space. But by Theorem 1.11 $\|F_x\| = \|\lambda(x)\|$.

Theorem 2.5: If λ is a quasi-kernel and $U \in \Delta$, then $\lambda(\cdot, U)$ is lower semicontinuous.

Proof: Using Corollary 1.12 we have $\lambda(x, U) = \sup\{\lambda(f)(x) : 0 \leq f \leq 1, f \in C_0(S) \text{ and } \text{spt}(f) \subset U\}$ for each $x \in T$. But since λ is a quasi-kernel, then $\lambda(f)$ is l.s.c. for all $f \geq 0$ in $C_0(S)$, and the supremum of l.s.c. functions is l.s.c.

Theorem 2.6: If λ is a quasi-kernel and $E \in \sigma(\Delta)$, then $\lambda(\cdot, E)$ is Borel measurable on T .

Proof: We let $\Sigma = \{E \in \sigma(\Delta) : \lambda(\cdot, E) \text{ is Borel measurable}\}$. Since the Borel algebra is generated by the open subsets of T then by Theorem 2.5 we have $\Sigma \supset \Delta$. If $A, B \in \Sigma$ and $A \supset B$ then, $\lambda(x, A \setminus B) = \lambda(x, A) - \lambda(x, B)$ for each $x \in T$ and therefore $A \setminus B \in \Sigma$. If $\{A_i : i=1, 2, \dots\} \subset \Sigma$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ then

$\lambda(x, \bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda(x, A_i)$ for each $x \in T$ and consequently $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.

By Corollary 2.2, $\Sigma = \sigma(\Delta)$, and the proof is complete.

Corollary 2.7: Let $\mu \in M(S)^+$, $k: T \times S \rightarrow R^+$, where R^+ denotes the set of non-negative real numbers. If the function $k(x, \cdot) \in L^1(S, \mu)$ for each $x \in T$ and $K_f(x) = \int_S k(x, y) f(y) \mu(dy)$ is l.s.c. for all $f \geq 0$ in $C_0(S)$, then $\int_E k(\cdot, y) \mu(dy)$ is Borel measurable for all Borel sets E .

We now consider an arbitrary kernel λ . It follows from our remarks in Chapter I that $\lambda(x) = [\lambda_1^+(x) - \lambda_1^-(x)] + i[\lambda_2^+(x) - \lambda_2^-(x)]$ where $\lambda_j^+(x), \lambda_j^-(x) \in M(S)^+$ for each $x \in T$ and $j = 1, 2$. Since the real and imaginary parts of λ also define kernels on T , it is evident that for the problem of measurability of $\lambda(\cdot, E)$ it suffices to consider real valued kernels.

Theorem 2.8: If λ is a real valued kernel and $\lambda(x) = \lambda^+(x) - \lambda^-(x)$, then λ^+ and λ^- are quasi-kernels and so is $|\lambda| = \lambda^+ + \lambda^-$.

Proof: Let f be a non-negative function in $C_0(S)$ and let $\mu(x, E) = \int_E f(y) \lambda(x, dy)$ for $x \in T$ and $E \in \sigma(\Delta)$. If $g \in C_0(S)$, then $\mu(g) = \lambda(fg)$ and hence $\mu(g)$ is continuous for $g \in C_0(S)$. Therefore μ is a kernel and furthermore $|\mu|(x, S) = \sup\{|\mu(g)(x)| : g \in C_0(S) \text{ and } \|g\| \leq 1\}$. Since each function $\mu(g)$ is continuous it follows that $|\mu|(\cdot, S)$ is l.s.c. But $|\mu|(x, S) = \int_S f(y) |\lambda|(x, dy) = \int_S f(y) \lambda^+(x, dy) + \int_S f(y) \lambda^-(x, dy) = \lambda^+(f)(x) + \lambda^-(f)(x)$. Hence for $f \geq 0$ we have $\lambda^+(f) + \lambda^-(f)$ is l.s.c. and since λ is a kernel $\lambda^+(f) - \lambda^-(f)$ is continuous.

It now follows that both $\lambda^+(f)$ and $\lambda^-(f)$ are l.s.c. for the sum of l.s.c. functions is l.s.c. Since $|\lambda| = \lambda^+ + \lambda^-$ this concludes the proof.

From Theorem 2.5 we then have

Corollary 2.9: If λ is a real-valued kernel, then $\lambda^+(\cdot, U)$, $\lambda^-(\cdot, U)$ and $|\lambda|(\cdot, U)$ are l.s.c. for each $U \in \Delta$.

In a later section we consider continuity of these same functions and relate this to the study of weakly compact operators on $C_0(S)$ and $C(S)_\beta$.

Theorem 2.10: If λ is a kernel and μ and ν are its real and imaginary parts, then (1) $\lambda(\cdot, E)$ is Borel measurable for each Borel set E ; (2) $\mu(\cdot, E)$, $|\mu|(\cdot, E)$, $\mu^+(\cdot, E)$ and $\mu^-(\cdot, E)$ are all Borel measurable; (3) same as (2) with μ replaced by ν .

For by Theorem 2.8 and Theorem 2.6 statements (2) and (3) hold since $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. But then (1) also holds.

Theorem 2.11: If λ is a quasi-kernel, then $\lambda(f) \in B(T)$ for $f \in B(S)$. Hence if λ is a kernel the same statement holds.

Proof: We let $f \in B(S)$, $f \geq 0$ and $E_k^n = \{s \in S : \frac{k-1}{n} \|f\| < f(s) \leq \frac{k}{n} \|f\|\}$ for $k = 1, 2, \dots, n$. Let $g_n = \sum_{k=1}^n \frac{k}{n} \|f\| \chi_{E_k^n}$. Then $0 \leq g_n - f \leq \|f\|/n$ and hence $\|\lambda(g_n) - \lambda(f)\|_\infty = \sup\{|\int [g_n(y) - f(y)] \lambda(x, dy)| : x \in T\} \leq \|g_n - f\| \sup\{\|\lambda(x)\| : x \in T\} \leq \|f\| \|\lambda\|/n$. Since $\|\lambda\| < \infty$ we would have $\lambda(f)$ measurable provided $\lambda(g_n)$ is for each n . But $\lambda(g_n)(x) = \sum_{k=1}^n \frac{k}{n} \|f\| \lambda(\cdot, E_k^n)$ and $\lambda(\cdot, E_k^n)$ is

measurable since $E_k^n \in \sigma(\Delta)$. This completes the proof for a quasi-kernel λ . If λ is any kernel then the positive and negative parts of its real and imaginary parts are all quasi-kernels and therefore the same conclusion holds for λ .

This concludes our study of the measurability of $\lambda(\cdot, E)$ for E Borel. Karel de Leeuw [24] points out that this problem is often slighted in the study of the convolution of measures on a locally compact group. It has been pointed out to the author that in the case of kernels of the type $\lambda(x, E) = \int_E k(x, y) \mu(dy)$, $k(x, \cdot) \in L^1(\mu)$ the problem is again often ignored. In many cases it is true that the space T is first countable and S is metrizable in which case the problem of measurability is simplified for the obvious reason that the limit of a sequence of measurable functions is again measurable. Dynkin [12] in his study of Markov processes and transition functions, where again this problem occurs, assumes that every open set in S is the set of non-zeroes of a continuous function on S thus making every open set an F_σ and consequently allowing one to choose a single sequence of closed sets $\{F_n\}$ which approximate a given open set in measure for all measures μ on S . In a general locally compact space S one can in general only obtain such a sequence $\{F_n\}$ for a given measure μ . It is this difficulty which is surmounted by our work above and which (see Chapter IV) will have interesting consequences.

We are now ready to consider transformations from the set of measures on T into $M(S)$ induced in a natural manner by a kernel λ .

Theorem 2.12: Let λ be a quasi-kernel and let μ be a bounded non-negative Borel measure on T . For each Borel set E , let $\nu(E) = \int_T \lambda(x, E) \mu(dx)$. Then ν is a bounded non-negative Borel measure and for $f \in B(S)$ we have $\int_S f d\nu = \int_T \lambda(f) d\mu$.

Proof: Since $\lambda(\cdot, E)$ is a bounded Borel function for each Borel set E then ν is defined for all Borel sets E . To see that ν is a Borel measure let $\{E_i\}$ be a countable collection of mutually disjoint Borel sets. Then $\lambda(x, \bigcup_{i=1}^{\infty} E_i)$
 $= \sum_{i=1}^{\infty} \lambda(x, E_i)$ for all $x \in T$ so that by Lebesgue's Monotone Convergence theorem we have

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \int_T \lambda(x, \bigcup_{i=1}^{\infty} E_i) \mu(dx) = \sum_{i=1}^{\infty} \int_T \lambda(x, E_i) \mu(dx) = \sum_{i=1}^{\infty} \nu(E_i).$$

Thus ν is a Borel measure and it is clear that ν is non-negative and bounded by $\|\lambda\| \|\mu\|$.

Now let $f \in B(S)$, $0 \leq f \leq 1$ and set $E_k^n = \{x \in S : \frac{k-1}{n} < f(x) \leq k/n\}$ and let $f_n = \sum_{k=1}^n \frac{k}{n} \chi_{E_k^n}$. Then $0 \leq f_n(x) - f(x) \leq \frac{1}{n}$ for all $x \in S$ and hence $\lim_{n \rightarrow \infty} \int_S f_n d\nu = \int_S f d\nu$ and also $0 \leq \lambda(f_n)(x) - \lambda(f)(x) \leq \frac{\|\lambda\|}{n}$ so that $\lim_{n \rightarrow \infty} \int_T \lambda(f_n) d\mu = \int_T \lambda(f) d\mu$. But
 $\int_T \lambda(f_n) d\mu = \int_T \sum_{k=1}^n \frac{k}{n} \lambda(x, E_k^n) \mu(dx) = \sum_{k=1}^n \frac{k}{n} \int_T \lambda(x, E_k^n) \mu(dx)$
 $= \sum_{k=1}^n \frac{k}{n} \nu(E_k^n) = \int_S f_n d\nu$. Therefore $\int_S f d\nu = \int_T \lambda(f) d\mu$. It follows that for any $f \in B(S)$, $\int_S f d\nu = \int_T \lambda(f) d\mu$ completing the proof.

If λ is an arbitrary kernel then due to Theorems 2.8 and 2.12 we have the following

Theorem 2.13: Let $\lambda: T \rightarrow M(S)$ be a kernel and μ a bounded Borel measure on T . Then the formula $\nu(E) = \int_T \lambda(x, E) \mu(dx)$ defines a bounded Borel measure on S such that $\int_S f d\nu = \int_T \lambda(f) d\mu$ for all $f \in B(S)$, and $|\nu|(E) \leq \int_T |\lambda|(x, E) |\mu|(dx)$ for all Borel sets E .

The reader will notice no doubt that we have not claimed that the measure ν is a regular measure. Our next task is to prove that ν is indeed a regular Borel measure. This result will lie at the foundation of all subsequent work in this paper and we will pause now to examine this problem and its importance.

The following result is a simple consequence of Corollary 2.2 and also can be found in Halmos [17].

Lemma 2.14: If α and β are two bounded Borel measures which are equal on all open subsets of S , then α and β agree on all Borel sets and are equal.

Consider now the functional $F(f) = \int_S f d\nu$ where ν is defined in Theorem 2.13 and $f \in C_0(S)$. Since ν is bounded, then F is a bounded linear functional on $C_0(S)$ and therefore by the Riesz theorem (Theorem 1.11), there is a measure $\omega \in M(S)$ such that $\int_S f d\nu = \int_S f d\omega$ for all $f \in C_0(S)$. Hence the Borel measures ν and ω agree on all functions in $C_0(S)$. If one could extend this equality to all open sets, then, by Lemma 2.14, ν would equal the regular measure ω . It should

be noted that certain authors avoid the problem by working with the measure ω , which since $\int_S f d\omega = \int_T \lambda(f) d\mu$ for all $f \in C_0(S)$, is the weak * integral $\int_T \lambda(x, E) \mu(dx)$. In his paper, "Existence of invariant measures for Markov processes," S. R. Foguel [14] is unable to avoid the problem and makes the assumption that the formula $\int_T P(t, x, E) \mu(dx)$ defines a regular measure, where $P(t, x, \cdot)$ is a transition function.

Let λ be a quasi-kernel. We now require that the space T also be a locally compact Hausdorff space and suppose $\mu \in M(T)^+$. We will show by a series of lemmas that $\nu(E) = \int_T \lambda(x, E) \mu(dx)$ is a regular Borel measure.

Lemma 2.15: Let U be an open subset of S , χ its characteristic function. Let $X = \{f \in C_0(S) : 0 \leq f \leq \chi\}$ and let $Y = \{g \in C_0(T) : 0 \leq g \leq \lambda(\cdot, U)\}$. Then

$$\sup\{\int_T g d\mu : g \in Y\} \leq \sup\{\int_T \lambda(f) d\mu : f \in X\}.$$

Proof: Let $g \in Y$, $\epsilon > 0$ and suppose g vanishes outside the compact set K and fix $x \in K$. Since $g \in Y$ then $g(x) - \epsilon/2 < \lambda(x, U)$ and by Corollary 1.12 there is a function $f_x \in X$ such that $g(x) - \epsilon/2 < \lambda(f_x)(x)$. Since $\lambda(f_x)$ is lower-semi-continuous there is a neighborhood V_x of x such that $g(x) - \epsilon/2 < \lambda(f_x)(t)$ for all $t \in V_x$. But also there is a neighborhood U_x of x such that $g(t) - \epsilon < g(x) - \epsilon/2$ for all $t \in U_x$. Then for $t \in W_x = U_x \cap V_x$ we have $g(t) - \epsilon < \lambda(f_x)(t)$. The collection of all sets W_x so obtained for each $x \in K$ is an open cover of the compact set K and so has a finite subcover $\{W_{x_i} : i=1, 2, \dots, n\}$.

Let $f(s) = \max\{f_{x_i}(s) : i=1,2,\dots,n\}$ for all $s \in S$. Then $f \in X$ and for $t \in K$ we have $g(t) - \varepsilon < \lambda(f)(t)$ since $t \in W_{x_i}$ for some i and $\lambda(f_{x_i}) \leq \lambda(f)$.

Hence, $\int_T g d\mu - \varepsilon \mu(T) \leq \int_T \lambda(f) d\mu$ completing the proof since $\mu(T) < \infty$.

Lemma 2.16: $\int_T \lambda(x, U) \mu(dx) \leq \sup\{\int_T g d\mu : g \in Y\}$.

Proof: Let $\varepsilon > 0$ and let n be an integer such that $n\varepsilon > \|\lambda\| \geq (n-1)\varepsilon$. Set $a_k = k\varepsilon$ and $E_k = \{x \in T : k\varepsilon < \lambda(x, U) \leq (k+1)\varepsilon\}$ for $k = 0, 1, 2, \dots, n-1$. Then $\{E_k : k=0, 1, \dots, n-1\}$ is a partition of T by Borel sets and

$$\begin{aligned} 0 &\leq \int_T \lambda(x, U) \mu(dx) - \sum_{k=0}^{n-1} a_k \mu(E_k) = \sum_{k=0}^{n-1} \int_{E_k} [\lambda(x, U) - a_k \chi_{E_k}(x)] \mu(dx) \\ &\leq \varepsilon \sum_{k=0}^{n-1} \mu(E_k) = \varepsilon \|\mu\|. \end{aligned}$$

Hence we have

$$(1) \quad 0 \leq \int_T \lambda(x, U) \mu(dx) - \sum_{k=0}^{n-1} a_k \mu(E_k) \leq \varepsilon \|\mu\|.$$

Let $U_k = \{x : \lambda(x, U) > k\varepsilon\}$. Then by Theorem 2.5, U_k is an open set and $E_k = U_k \setminus U_{k+1}$, $k = 0, 1, \dots, n-1$.

Since μ is assumed to be a regular measure on T there then exists for each $k = 0, 1, \dots, n-1$ a compact set $K_k \subset E_k$ such that $\mu(E_k \setminus K_k) < \varepsilon/n^2$. Furthermore, since $K_k \subset U_k$ and T is locally compact there exists for each k an open set V_k with compact closure such that $K_k \subset V_k \subset \bar{V}_k \subset U_k$. There then exist functions $f_k \in C_c(T)^+$ for $k = 0, 1, \dots, n-1$ such that $f_k(x) = a_k$ for $x \in K_k$ and $f_k(x) = 0$ for $x \in T \setminus V_k$ and $\|f_k\| \leq a_k$.

Let $x \in \bar{V}_k$. Then $f_k(x) \leq a_k = k\varepsilon < \lambda(x, U)$ since $\bar{V}_k \subset U_k$. But $f_k \equiv 0$ outside \bar{V}_k and therefore $f_k \in Y$.

Now let $f(x) = \max \{f_k(x) : k=0, 1, \dots, n-1\}$. We claim $f \in Y$. For $x \in T$ implies $x \in E_k$ for some k and hence $x \in U_k \setminus U_{k+1}$. Since $U_{n-1} \subset U_{n-2} \subset \dots \subset U_1 \subset U_0$ this means $x \in \bigcap_{j=0}^k U_j$ and $x \notin \bigcup_{j=k+1}^{n-1} U_j$. Thus $f_j(x) = 0$ for $j \geq k+1$ and $f_j(x) \leq a_j$ for $j \leq k$. But if $j \leq k$ then $a_j \leq a_k$ and $\lambda(x, U) > a_k$ since $x \in U_k$. Therefore $f(x) < \lambda(x, U)$ and $f \in Y$.

This argument also shows $f(x) \leq \sum_{k=0}^{n-1} a_k \chi_{E_k}(x)$ so that

$$\begin{aligned} 0 &\leq \int_T \sum_{k=0}^{n-1} a_k \chi_{E_k} d\mu - \int_T f d\mu \\ &\leq \sum_{k=0}^{n-1} \int_{E_k} (a_k - f) d\mu \\ &\leq \sum_{k=0}^{n-1} \int_{E_k} (a_k - f_k) d\mu \\ &= \sum_{k=0}^{n-1} \int_{E_k \setminus K_k} (a_k - f_k) d\mu \\ &\leq \sum_{k=0}^{n-1} \int_{E_k \setminus K_k} a_k d\mu \\ &\leq \sum_{k=0}^{n-1} a_k \mu(E_k \setminus K_k) \\ &\leq n\varepsilon \sum_{k=0}^{n-1} \varepsilon/n^2 = \varepsilon^2. \end{aligned}$$

Hence, (2) $0 \leq \int_T \sum_{k=0}^{n-1} a_k \chi_{E_k} d\mu - \int_T f d\mu < \varepsilon^2$ and combining (1)

and (2) yields $0 \leq \int_T \lambda(x, U) \mu(dx) - \int_T f d\mu < \varepsilon \|\mu\| + \varepsilon^2$ since

$$\int_T \sum_{k=0}^{n-1} a_k \chi_{E_k} d\mu = \sum_{k=0}^{n-1} a_k \mu(E_k). \text{ Since } f \in Y \text{ this means}$$

$$\int_T \lambda(x, U) \mu(dx) + \varepsilon \|\mu\| + \varepsilon^2 < \sup \{ \int_T g d\mu : g \in Y \} \text{ for all } \varepsilon > 0,$$

yielding the desired conclusion.

Lemma 2.17: $\nu(U) = \sup\{\int_S f d\nu : f \in X\}$ and consequently ν is regular.

Proof: Combining Lemmas 2.15 and 2.16 we have

$$\begin{aligned}\nu(U) &= \int_T \lambda(x, U) \mu(dx) \\ &= \sup\{\int_T g d\mu : g \in Y\} \\ &= \sup\{\int_T \lambda(f) d\mu : f \in X\} \\ &= \sup\{\int_S f d\nu : f \in X\}\end{aligned}$$

this last equality following from Theorem 2.12. But if $f \in X$, then since ν is positive, $\nu(U) \geq \int_S f d\nu$. Hence

$$\nu(U) = \sup\{\int_S f d\nu : 0 \leq f \leq \chi, f \in C_c(S)\}.$$

From our remarks following Lemma 2.14 there is a regular measure $\omega \geq 0$ such that $\int_S f d\nu = \int_S f d\omega$ for all $f \in C_0(S)$. But since ω is regular we have $\omega(U) = \sup\{\int_S f d\omega : 0 \leq f \leq \chi, f \in C_c(S)\} = \nu(U)$. This holds for all open sets U and so by Lemma 2.14, ν is the regular measure ω .

We have now proven

Theorem 2.18: Let S and T be locally compact Hausdorff spaces. Let $\lambda : T \rightarrow M(S)$ be a quasi-kernel and $\mu \in M(T)^+$. Let $\nu(E) = \int_T \lambda(x, E) \mu(dx)$ for all Borel sets E . Then $\nu \in M(S)^+$.

If μ is an arbitrary member of $M(T)$ then $\mu = (\alpha^+ - \alpha^-) + i(\beta^+ - \beta^-)$ where α^\pm and β^\pm belong to $M(T)^+$. It follows from the result above that $\nu(E) = \int_T \lambda(x, E) \mu(dx)$ belongs to $M(S)$ for the quasi-kernel λ . Combining this with Theorem 2.8 applied to the real and imaginary parts of an arbitrary

kernel λ we have

Theorem 2.19: If $\lambda: T \rightarrow M(S)$ is a kernel with T locally compact and Hausdorff and if $\mu \in M(T)$ then $\nu(E) = \int_T \lambda(x, E) \mu(dx)$ is a bounded regular Borel measure on S . If λ is a real valued kernel then so is the measure $\omega(E) = \int_T |\lambda|(x, E) \mu(dx)$ a member of $M(S)$.

We conclude this section with an application to the integral representation of operators on the space $C_0(S)$. Our theorems constitute an improvement on, and slight generalization of, the results found in Dunford and Schwartz [9, p.490].

Theorem 2.20: Let A be linear operator from $C_0(S)$ into $C(T)$. If A is continuous from $C_0(S)$ with the weak topology into $C(T)$ with the topology of pointwise convergence on T , then there is a unique kernel $\lambda: T \rightarrow M(S)$ such that $A(f) = \lambda(f)$ for all $f \in C_0(S)$ and A is a bounded operator. Conversely, if $\lambda: T \rightarrow M(S)$ is a kernel, then the formula $A(f) = \lambda(f)$ defines a bounded linear operator on $C_0(S)$ into $C(T)$.

Proof: For each $x \in T$ the functional $F_x(f) = [Af](x)$ is weakly continuous on $C_0(S)$ and so by the Riesz Representation Theorem there is a unique measure $\lambda(x) \in M(S)$ such that $F_x(f) = \int_S f(y) \lambda(x, dy) = \lambda(f)(x)$. Hence $Af = \lambda(f)$ for all $f \in C_0(S)$ and λ is unique. Furthermore $\sup\{\|Af\|: f \in C_0(S), \|f\| \leq 1\} = \sup\{\|\lambda(f)\|: f \in C_0(S), \|f\| \leq 1\} \leq \|\lambda\| < \infty$ by Lemma 2.4 so that A is a bounded operator. Conversely it is clear from Lemma 2.4

that the formula $Af = \lambda(f)$ defines a bounded linear operator on $C_0(S)$.

Before proceeding further let us consider the dual space of $C(T)$ with the norm topology. Using Theorem 1.11 it is not difficult to see that $C(T)^* = M(\beta T)$ where βT is the Stone-Čech compactification of T . For our purposes it is enough to notice that we can embed $M(T)$ in Y by means of the mapping $\mu \rightarrow F\mu$ where $\mu \in M(T)$ and $F\mu(f) = \int_T f d\mu$. For certainly $F\mu$ is a bounded linear functional on $C(T)$ and $\|F\mu\| = \sup\{|\int_T f d\mu| : \|f\| \leq 1\} = \|\mu\|$. In the statement of our next theorem we do not distinguish between μ and $F\mu$ and consider $M(T)$ as a subset of $C(T)^*$.

Theorem 2.21: If the linear operator A and the kernel λ are related as in Theorem 2.20, then

- (1) $\|A\| = \|A^*\| = \|\lambda\|$.
- (2) $\lambda(x) = A^* \overset{\circ}{x}$ for all $x \in T$ where $\overset{\circ}{x}$ is the point measure concentrated at x .
- (3) $A^*(M(T)) \subset M(S)$, and for $\mu \in M(T)$ we have $(A^*\mu)(E) = \int_T \lambda(x, E) \mu(dx)$.
- (4) For all $f \in B(S)$ we have $A^{**}f \in B(T)$ and $A^{**}f = \lambda(f)$, where A^{**} is the adjoint of the restriction of A^* to $M(T)$.
- (5) $\lambda(x, E) = (A^{**}\chi_E)(x)$ for all $x \in T$ and all Borel sets E .

Proof: The proof of Theorem 2.20 shows $\|A\| \leq \|\lambda\|$. If $\epsilon > 0$ then there is an x such that $\|\lambda\| - \epsilon < \|\lambda(x)\|$ and there exists $f \in C_0(S)$, $\|f\| \leq 1$ such that $\|\lambda\| - \epsilon < |\lambda(f)(x)| \leq \|\lambda(x)\|$. But $|\lambda(f)(x)| = |(Af)(x)| \leq \|A\|$ and hence $\|\lambda\| = \|A\|$. From

Theorem 1.5 A^* is continuous from $C(T)^*$ into $M(S)$ and hence is bounded since $C(T)^*$ is a Banach space. From the well known result in [9, p.478] $\|A\| = \|A^*\|$. If $\mu \in M(T) \subset C(T)^*$ in the sense defined above we have for all $f \in C_0(S)$ $\langle f, A^*\mu \rangle = \langle Af, \mu \rangle = \int_T Af \, d\mu = \int_T \lambda(f) \, d\mu$. By Theorem 2.13 the measure $\nu(E) = \int_T \lambda(x, E) \, \mu(dx)$ has the property $\int_S f \, d\nu = \int_T \lambda(f) \, d\mu$ for all $f \in C_0(S)$ so that $\langle f, A^*\mu \rangle = \int_S f \, d\nu = \langle f, \nu \rangle$. Since ν is regular (i.e., $\nu \in M(S)$) and this holds for all $f \in C_0(S)$ we must have $\nu = A^*\mu$ proving (3). Using (3) with $\mu = \overset{\circ}{x}$ we get $(A^* \overset{\circ}{x})(E) = \int_T \lambda(y, E) \overset{\circ}{x}(dy) = \lambda(x, E)$ so that $A^* \overset{\circ}{x} = \lambda(x)$ and (2) holds.

To obtain (4) we consider $B(S)$ and $B(T)$ as imbedded in $M(S)^*$ and $M(T)^*$ respectively in the sense of our remarks following Corollary 1.12. If $\mu \in M(T)$ and $f \in B(S)$, then by (2) and Theorem 2.13 we have $\langle A^{**}f, \mu \rangle = \langle f, A^*\mu \rangle = \int_S f \, dA^*\mu = \int_T \lambda(f) \, d\mu = \langle \lambda(f), \mu \rangle$. Since $M(T)$ separates points of $M(T)^*$ this means $A^{**}f = \lambda(f) \in B(T)$ by Theorem 2.11 and (4) holds. From (4) follows (5) with $f = \chi_E$ for then $\lambda(x, E) = \lambda(f)(x)$.

Corollary 2.2: Let B be a linear operator from $M(T)$ to $M(S)$ and suppose $B^*(C_0(S)) \subset C(T)$. Then B is bounded and (1) $B^*(B(S)) \subset B(T)$ and $[B^*f](x) = \int_S f \, d\overset{\circ}{Bx}$; (2) for all $\mu \in M(T)$, $(B\mu)(E) = \int_T (\overset{\circ}{Bx})(E) \, \mu(dx)$.

Proof: Let $\lambda(x, E) = (\overset{\circ}{Bx})(E) = \langle \chi_E, \overset{\circ}{Bx} \rangle = \langle B^*\chi_E, \overset{\circ}{x} \rangle$ and let $A = B^*|_{C_0(S)}$. Then for $f \in C_0(S)$ we have $\lambda(f)(x) = \int_S f \, d\overset{\circ}{Bx} = \int_S B^*f \, d\overset{\circ}{x} = \int_S Af \, d\overset{\circ}{x} = [Af](x)$ so that λ is the kernel for A . Furthermore if $\mu \in M(T)$ and $f \in C_0(S)$, we have $\langle f, A^*\mu \rangle = \langle Af, \mu \rangle = \langle B^*f, \mu \rangle = \langle f, B\mu \rangle$ so that $B = A^*|_{M(T)}$. Then (3) and (4)

of Theorem 2.21 imply (3) and (2) above respectively and the boundedness of B follows from Lemma 2.4 and $B^*(C_0(S)) \subset C(T)$.

As we noted in Chapter I the collection of unimodular multiples of the point measure on T is exactly the set of extremal points of the unit sphere in $M(T)$. The set of point measures $\{\overset{\circ}{x}: x \in T\}$ also is a basis for $M(T)$ in the sense that of $\mu \in M(T)$ then $\mu(E) = \int_S \chi_E d\mu = \int_S \overset{\circ}{x}(E) \mu(dx)$ since $\chi_E(x) = \overset{\circ}{x}(E)$. Theorem 2.21 and Corollary 2.22 say that the operators A and B are completely determined by their action on the point measures since, for example, $(A^*\mu)(E) = \int_T (A^* \overset{\circ}{x})(E) \mu(dx)$ from (2) and (3) of Theorem 2.21.

Further observations can be made. Considering (4) of Theorem 2.21 we see that an operator A on $C_0(S)$ into $C(T)$ has a natural extension $\bar{A} = A^{**}$ to $B(S)$ into $B(T)$. We ask the natural questions: When is it true that $\bar{A}(C(S)) \subset C(T)$? When is it true that $\bar{A}(B(S)) \subset C(T)$? When is \bar{A} unique? In succeeding chapters we will consider these questions and also see surprising applications of our theory as a consequence.

The interested reader may refer to Dunford and Schwartz [9, p.492], and using Theorem 2.21, see that the vector-valued measure representation obtained there is none other than $\mu(E) = \lambda(\cdot, E) \in B(T)$ for all Borel sets E . This represents an improvement on the result obtained there in that the range of μ can be restricted to $B(T)$ rather than $C(T)^{**}$.

Before considering any further topics we review certain of our results. First of all one may wish to consider mappings

$\lambda: T \rightarrow M(S)$ such that $\lambda(f) \in C(T)$ for $f \in C_c(S)$. But it is easily seen that $\|\lambda\| < \infty$ if and only if $\lambda(f) \in C(T)$ for $f \in C_0(S)$ since $C_c(S)$ is dense in $C_0(S)$.

Finally, Theorem 2.21 is crucial to our development. While certain parts of its proof may be simplified using the general theory of locally convex vector spaces one sees that the interesting results (2) and (4) depend solely on Theorem 2.19 --that is, that ν is a regular measure. On a locally compact metrizable space all bounded Borel measures are regular; this follows from results found in Halmos [17]. An example of a non-regular measure can also be found in Halmos [17, p.231].

In closing this chapter, we point out an application to integration on locally compact groups or compact semigroups G . If μ is a bounded regular Borel measure on G , set $\lambda(x, E) = \mu(Ex^{-1})$ where $Ex^{-1} = \{x' \in G: x'x \in E\}$. Let $[Af](x) = \lambda(f)(x) = \int_G f(yx) \mu(dy)$. Then A has range $C(G)$, and so by Theorem 2.11, $\lambda(x, E) = \mu(Ex^{-1})$ is measurable on G . Furthermore, $(A^*\nu)(E) = \int_G \lambda(x, E) \nu(dx) = \int_G \mu(Ex^{-1}) \nu(dx)$, while also, $\int_G \int_G f(yx) \mu(dy) \nu(dx) = \int_G \lambda(f)(x) \nu(dx) = \int_G f dA^*\nu$ and this equals $\int_G f d(\mu * \nu)$, where $\mu * \nu$ is the convolution of μ with ν . We have thus shown that $(\mu * \nu)(E) = \int_G \mu(Ex^{-1}) \nu(dx) = (A^*\nu)(E)$. Thus our Theorem 2.21 yields certain basic results found in [18] and [24] as a special case.

CHAPTER III

KERNELS AND OPERATORS ON $C(S)_\beta$

In this chapter we extend the results of Chapter II to the spaces $C(S)_\beta$ and $C(S)_{\beta'}$. We also determine necessary and sufficient conditions on the measures $\{\lambda(x): x \in T\}$ so that $\lambda(f) \in C(T)$ for all $f \in C(S)$ when S is paracompact.

Definition 3.1: Let $\lambda: T \rightarrow M(S)$. We say that λ satisfies condition E (e') if $\{\lambda(x): x \in K\}$ is β -equicontinuous (β' -equicontinuous) for all compact sets $K \subset T$.

Since the β' -topology on $C(S)$ is finer than the β topology it follows that condition E' is weaker than condition E. From Conway's characterization of β -equicontinuity (see (b) of Theorem 1.15) we see that λ satisfies condition E if and only if for each compact set $K \subset T$ the set $\{\lambda(x): x \in K\}$ is uniformly bounded and for each $\epsilon > 0$ there is a compact set $Q \subset S$ such that $|\lambda|(x, S \setminus Q) < \epsilon$ for all $x \in K$.

Theorem 3.1: If $\lambda: T \rightarrow M(S)$ is a kernel which satisfies condition E' then $\lambda(f) \in C(T)$ for all $f \in C(S)$.

Proof: By Lemma 2.4, $\lambda(f)$ is a bounded function on T for all $f \in C(S)$. Hence it remains to show $\lambda(f)$ is continuous on T . To show this let $x \in T$ and $f \in C(S)$. Let U be a neighborhood of x with compact closure. Since λ satisfies condition E' then $\{\lambda(x): x \in U\}$ is β' -equicontinuous. Hence there is a β' -neighborhood of 0, V , in $C(S)$ such that for all $g \in V$ one has $|\lambda(g)(y)| \leq 1$ for all $y \in U$. We claim we can choose $g \in C_0(S)$ such that $f - g \in \alpha V$ for a given $\alpha > 0$. For let $r > 2\|f\|$. Since

αV is a β' neighborhood of 0 there is a β -neighborhood of 0, W , such that $B_r \cap W \subset \alpha V$. We can suppose $W \supset V_\phi$, $\phi \geq 0$, $\phi \in C_0(S)$ where $V_\phi = \{h \in C(S) : \|\phi h\| \leq 1\}$. Then the set $Q = \{x : \phi(x) \geq 1/2\|f\|\}$ is compact and there is a function $\psi \in C_0(S)$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on Q . Then $\psi f \in C_0(S)$ and for $x \in Q$ one has $|\phi(x)f(x) - \phi(x)\psi(x)f(x)| = 0$ and for $x \notin Q$ one has $|\phi(x)f(x) - \phi(x)\psi(x)f(x)| = |\phi(x)| |f(x) - \psi(x)f(x)| \leq \phi(x) 2\|f\| \leq 1$ so that $f - \psi f \in V_\phi$ and further $\|f - \psi f\| \leq 2\|f\| < r$ so that $f - g \in B_r \cap V_\phi \subset \alpha V$ where $g = \psi f \in C_0(S)$. Hence $(f-g)/\alpha \in V$ and consequently $|\lambda(f)(y) - \lambda(g)(y)| \leq \alpha$ for all $y \in U$. But since λ is a kernel and $g \in C_0(S)$ then $\lambda(g) \in C(T)$ and hence $\lambda(f)$ is the uniform limit of continuous functions on U and must be continuous.

The reader will notice that essentially the proof above hinged on showing that $C_0(S)$ is β' dense in $C(S)$ and concluding that of an equicontinuous collection of linear functionals (the functionals defined by $f \mapsto \lambda(f)(x)$) converges on a dense subset of a linear space then convergence holds on the entire space.

Using Theorem 1.16 due to Conway, we obtain a partial converse to Theorem 3.1.

Theorem 3.2: Suppose S is paracompact or $\lambda(x) \in M(S)^+$ for all $x \in T$ and that $\lambda(f) \in C(T)$ for all $f \in C(S)$. Then λ satisfies condition E.

Proof: Since S is paracompact it suffices to show, using Theorem 1.16 and the remarks following it, that $\{\lambda(x) : x \in K\} = A_K$

is β -weak* compact for each compact set K in T . Let $\{\lambda(x_\alpha)\}$ be a net in A_K . Then $\{x_\alpha\}$ is a net in K and so, since K is compact, has a cluster point $x \in K$. Hence for $f \in C(S)$, since $\lambda(f) \in C(T)$, we have that $\lambda(f)(x_\alpha) = \int_S f(y) \lambda(x_\alpha, dy)$ clusters to $\lambda(f)(x) = \int_S f(y) \lambda(x, dy)$. But this is exactly the statement that $\{\lambda(x_\alpha)\}$ clusters β -weak* to $\lambda(x)$. Hence every net in A_K has a β -weak* cluster point and A_K must be β -weak* compact and hence β -equicontinuous.

Notice that when S is paracompact the β and β' topologies coincide (see Theorem 1.17(e)) and therefore conditions E and E' coincide.

We pause now and consider two interesting kernels λ and show how our results above are generalizations of known results about β -weak* convergence in $M(S)$.

For our first example we suppose $\{\mu_n\}$ is a sequence in $M(S)$ which converges weak* (i.e. $\sigma(M(S), C_0(S))$) to the measure μ . We let T be the one point compactification of the integers with the discrete topology, with ω the point at infinity, and set $\lambda(n, E) = \mu_n(E)$, $\lambda(\omega, E) = \mu(E)$.

Then if $f \in C_0(S)$ we have $\lambda(f)(n) = \int_S f(y) \lambda(n, dy) = \int_S f d\mu_n$, and $\lambda(f)(\omega) = \int_S f d\mu$, so that $\lambda(f)$ is continuous on T . This also says the collection $\{\mu_n\}$ is pointwise bounded on $C_0(S)$ and so is uniformly bounded. This all means λ is a kernel on T into $M(S)$. Since the topology on T is discrete save at infinity we have the following known corollary of Theorems 3.1 and 3.2.

Corollary 3.3: If S is paracompact and $\{\mu_n\}$ is a sequence in $M(S)$ then $\mu_n \rightarrow \mu$ β -weak' if and only if $\mu_n \rightarrow \mu$ $\sigma(M(S), C_0(S))$ and $\{\mu_n\}$ is β -equicontinuous.

Our second example generalizes the above example and shows how our theory may be used to handle nets $\{\nu_\alpha\}$ in $M(S)$.

We consider a closed and bounded set M in $M(S)$. By Theorem 1.3 the set M with the topology $\sigma(M(S), C_0(S))$ is compact. We let T be M with the weak* topology and for $\nu \in T$ set $\lambda(\nu, E) = \nu(E)$. We claim λ is a kernel on T into $M(S)$. For if $f \in C_0(S)$ then $\lambda(f)(\nu) = \int_S f(y) \lambda(\nu, dy) = \int_S f d\nu$ for all $\nu \in T$. But with the weak* topology on M this makes the function $\lambda(f)$ continuous and bounded on T since M is bounded.

Suppose now that $\{\nu_\alpha\}$ is a bounded net in $M(S)$ which converges weak* to a measure ν . Then $\{\nu_\alpha\}, \{\nu\} \subset M$ for some closed and bounded set M and our hypothesis says $\nu_\alpha \rightarrow \nu$ in the topology on T so that since λ is a kernel then $\lambda(f)(\nu_\alpha) \rightarrow \lambda(f)(\nu)$ (or $\int f d\nu_\alpha \rightarrow \int f d\nu$) for all $f \in C(S)$ if and only if $\{\nu_\alpha\}$ is β -equicontinuous when S is paracompact.

We now turn to operator theoretic considerations. Beginning with a kernel λ on $C_0(S)$ we have given conditions on λ so that λ maps $C(S)$ into $C(T)$ under the natural mapping $f \rightarrow \lambda(f)$. We will call such a kernel a continuous kernel since it preserves continuity. Such a kernel defines a linear operator A on $C(S)$ into $C(T)$ by means of the formula $Af = \lambda(f)$. It is clear that A is continuous with the norm topologies on both these spaces. This leaves something to be desired however

in that not every norm continuous operator on these spaces has a kernel representation (see the example preceeding Theorem 5.5). Furthermore A can be considered as an extension of the operator $f \mapsto \lambda(f)$, defined for $f \in C_0(S)$, to $C(S)$ and norm continuity does not guarantee uniqueness of this extension. We will show that the β and β' topologies remove these deficiencies and, what is more, that no further conditions, other than conditions E or E' , need be placed on the kernel λ to gain these results. This means, due to Theorems 3.1 and 3.2, that we need no further condition on λ but that $\lambda(f) \in C(T)$ for all $f \in C(S)$ (i.e., that λ be a continuous kernel) to gain our results when S is paracompact.

For the sake of completeness we begin with the well-known

Lemma 3.4: Let E and F be locally convex Hausdorff topological vector spaces and A a weakly continuous linear operator on E into F . Then A is continuous if and only if A^* takes equicontinuous sets of F^* into equicontinuous sets of E^* .

Proof: By Theorem 1.5, $A^*(F^*) \subset E^*$ since A is weakly continuous. If A is a continuous operator and B is an equicontinuous set in F^* then $B \subset V^0$ where V is a neighborhood of 0 in F . Hence there is a neighborhood U of 0 in E such that $U \subset A^{-1}(V)$ or $A(U) \subset V$ and consequently $A(U)^0 \supset V^0$. But as is easily seen $A(U)^0 = A^{*-1}(U^0)$ so that $U^0 \supset A^*(V^0) \supset A^*(B)$ and by definition $A^*(B)$ is equicontinuous. Conversely if V is a closed absolutely convex 0-neighborhood in F then $A^*(V^0)$ being

equicontinuous means there is a 0-neighborhood U in E such that $U^0 \supset A^*(V^0)$ and consequently $V^0 \subset A^{*-1}(U^0) = A(U)^0$ and this says $V \supset A(U)$ and hence $A^{-1}(V) \supset U$ proving that A is continuous.

We now obtain our first representation theorem for operators on $C(S)_\beta$.

Theorem 3.5: Let A be a continuous linear operator from $C(S)_\beta, (C(S)_\beta,)$ into X where X denotes any one of the spaces $C(T)_\beta, C(T)_\beta,$ or $(C(T), \|\cdot\|)$. Then A is uniquely representable by a continuous kernel $\lambda: T \rightarrow M(S)$ such that $\lambda(f) = A(f)$ for all $f \in C(S)$. Furthermore λ satisfies condition $E(E')$.

Proof: From Theorem 1.14, $C(S)_\beta^* = M(S)$ and from this same result and our remarks preceding Theorem 2.21 we have $M(T) \subset X^*$. For $x \in T$ set $\lambda(x) = A^* \overset{0}{x} \in M(S)$ by Theorem 1.5. Then $[Af](x) = \int_T Af \, d\overset{0}{x} = \langle Af, \overset{0}{x} \rangle = \langle f, A^* \overset{0}{x} \rangle = \int_S f \, dA^* \overset{0}{x} = \lambda(f)(x)$ for all $f \in C(S)$. Clearly λ is unique and since A has range $C(T)$ λ is a continuous kernel. Finally if K is a compact subset of T then $B = \{\overset{0}{x} : x \in K\}$ is equicontinuous in X^* . For if $X = C(T)_\beta$ (or $C(T)_\beta,$) then $B \subset V_\phi^0$ where ϕ is a non-negative function in $C_0(T)$ which is identically 1 on K . If $X = (C(T), \|\cdot\|)$ then B is contained in U^0 where U is the unit ball in $C(T)$. In either case, by Lemma 3.4, $A^*(B) = \{\lambda(x) : x \in K\}$ is β -equicontinuous (β' -equicontinuous) when A is continuous on $C(S)_\beta$ ($C(S)_{\beta'}$) completing the proof.

If $X = (C(T), \|\cdot\|)$ we can obtain (see Theorem 3.12) a stronger condition than E . We now state and prove the analogue of Theorem 2.21 for β or β' continuous operators on $C(S)$.

Theorem 3.6: If the linear operator A on $C(S)_\beta$ or $C(S)_{\beta'}$ into $C(T)_\beta$ or $C(T)_{\beta'}$, or $(C(T), \|\cdot\|)$ is continuous and represented by the continuous kernel λ then

- (1) For all $\mu \in M(T)$ we have $(A^*\mu)(E) = \int_T \lambda(x, E) \mu(dx)$ for all Borel sets E .
- (2) $\lambda(x) = A^*x_0$.
- (3) A^{**} takes $B(S)$ into $B(T)$ and for all $f \in B(S)$, $A^{**}f = \lambda(f)$ where A^{**} is the adjoint of the restriction of A^* to $M(T)$.
- (4) $(A^{**}\chi_E)(x) = \lambda(x, E)$.
- (5) $\|A^*\| = \|A^{**}\| = \|\lambda\| = \sup\{\|Af\| : f \in C(S), \|f\| \leq 1\}$.

Proof: The proof is exactly the same as that of Theorem 2.21. Again using the fact that the integral on the right in (1) defines a regular measure ν such that $\int_S f d\nu = \int_T \lambda(f) d\mu = \int_T Afd\mu = \int_S f dA^*\mu$ for all $f \in C(S)$ we obtain $A^*\mu = \nu$ and (1) holds and from which follows (2). As in Theorem 2.21 we obtain (3) and (4). Using (1) we easily obtain $\|A^*\| = \|\lambda\|$ and (3) gives $\|A^{**}\| = \|\lambda\|$. Clearly since $Af = \lambda(f)$ for all $f \in C(S)$ we obtain $\|\lambda\| = \sup\{\|Af\| : f \in C(S), \|f\| \leq 1\}$.

We now obtain the converse of Theorem 3.5 for the β' topology on $C(S)$.

Theorem 3.7: Let $\lambda: T \rightarrow M(S)$ be a kernel satisfying condition E' . Then λ is a continuous kernel and the formula $Af = \lambda(f)$ defines a continuous linear operator from $C(S)_\beta$ into $C(T)_\beta$.

Proof: We know that λ is a continuous kernel from Theorem 3.1 so that $A(C(S)) \subset C(T)$.

To show that A is a continuous operator let V be a β' neighborhood of 0 in $C(T)$ and let $r > 0$. We show there is a β' -neighborhood of 0, W in $C(S)$ such that $A^{-1}(V) \supset B_r \cap W$ thus showing $A^{-1}(V)$ is a β' neighborhood of 0 in $C(S)$ and consequently proving continuity of A .

Let $P = r\|\lambda\|$. Since V is a β' neighborhood there is a function $\phi \in C_0(T), \phi \geq 0$ such that $V \supset B_P \cap V_\phi$, where $V_\phi = \{g \in C(T) : \|g\phi\| \leq 1\}$.

Let $K = \{t \in T : \phi(t) \geq 1/(P+1)\}$. Then K is compact and since λ satisfies condition E' there is a β' -neighborhood of 0, say U , such that $\{\lambda(x) : x \in K\} \subset U^0$; i.e., for all $x \in K$ and $f \in U$ we have $|\lambda(f)(x)| \leq 1$.

Let $W = \{f \in C(S) : \|\phi\|f \in U\}$. Then $A^{-1}(V) \supset B_r \cap W$ for if $f \in B_r \cap W$ then $\|f\| \leq r$ and hence $\|Af\| \leq \|\lambda\| \|f\| \leq P$ using (5) of Theorem 3.6, so that $Af \in B_P$ and furthermore $\|\phi Af\| \leq 1$ or $Af \in V_\phi$. For if $x \notin K$, then $|\phi(x)[Af](x)| \leq \frac{1}{P+1} \|Af\| \leq \frac{P}{P+1} \leq 1$ and if $x \in K$, then $|\phi(x)[Af](x)| \leq \|\phi\| |[Af](x)| = |A[\|\phi\|f](x)| = |\lambda(\|\phi\|f)(x)| \leq 1$ since $\|\phi\|f \in U$ and $x \in K$.

Therefore $\|\phi Af\| \leq 1$ and consequently $Af \in B_P \cap V_\phi$ for all $f \in B_r \cap W$. Hence since $V \supset B_P \cap V_\phi$ this means $A^{-1}(V) \supset B_r \cap W$, completing the proof.

Using Theorem 3.7 we now obtain a partial converse to Theorem 3.5 for the β topology.

Theorem 3.8: Suppose $\lambda: T \rightarrow M(S)$ is a kernel which satisfies condition E. Then λ is a continuous kernel and the formula $Af = \lambda(f)$ defines a continuous linear operator from $C(S)_\beta$ into $C(T)_\beta$.

Proof: Since λ satisfies condition E and β' is a finer topology than β then λ satisfies condition E' and consequently A is continuous on $C(S)_{\beta'}$ to $C(T)_{\beta'}$. In particular using (1) of Theorem 3.6 we have $(A^*\mu)(E) = \int_T \lambda(x, E) \mu(dx)$. By Lemma 3.4 it suffices to show that A^* takes β -equicontinuous sets into β -equicontinuous sets. To prove this we make use of Conway's characterization of β -equicontinuity (Theorem 1.15(b)). Let H be a β -equicontinuous set in $M(T)$. We show $A^*(H)$ is β -equicontinuous. Let $\epsilon > 0$ and let $\alpha = \sup\{\|\mu\| : \mu \in H\}$. Since H is β -equicontinuous there is a compact set $Q \subset T$ such that $|\mu|(T \setminus Q) < \epsilon/2\|\lambda\|$ for all $\mu \in H$. Since λ satisfies condition E the set $\{\lambda(x) : x \in Q\}$ is β -equicontinuous in $M(S)$ and so there is a compact set $P \subset S$ such that $|\lambda|(x, S \setminus P) < \epsilon/2\alpha$ for all $x \in Q$. We then have for all $\mu \in H$ that $|A^*\mu|(S \setminus P) < \int_T |\lambda|(x, S \setminus P) |\mu|(dx) \leq \int_Q |\lambda|(x, S \setminus P) |\mu|(dx) + \int_{T \setminus Q} |\lambda|(x, S \setminus P) |\mu|(dx) \leq \int_Q \epsilon/2\alpha |\mu|(dx) + \|\lambda\| |\mu|(T \setminus Q) \leq \epsilon/2\alpha \|\mu\| + \|\lambda\| \epsilon/2\|\lambda\| = \epsilon$. Since it is clear that $A^*(H)$ is uniformly bounded then from (a) and (b) of Theorem 1.15, $A^*(H)$ is β -equicontinuous completing the proof.

If we restrict ourselves to a paracompact space S we can nicely summarize our results as follows.

Theorem 3.9: If S is paracompact and A is a weakly continuous linear operator from $C(S)_\beta$ into $C(T)$ with the topology of pointwise convergence on T , then A is given uniquely by a continuous kernel λ and consequently is continuous on $C(S)_\beta$ into $C(T)_\beta$. Conversely, if λ is a continuous kernel on T into $M(S)$, then the formula $Af = \lambda(f)$ for $f \in C(S)$ defines a continuous linear operator on $C(S)_\beta$ into $C(T)_\beta$.

Proof: By hypothesis for each $x \in T$ the mapping $f \rightarrow [Af](x)$ is a weakly continuous linear functional on $C(S)_\beta$ and hence is given by a unique measure $\lambda(x) \in M(S)$. The mapping λ so defined for all $x \in T$ is a continuous kernel since $\lambda(f)(x) = [Af](x)$ for all $f \in C(S)$ and $x \in T$ and $Af \in C(T)$. By Theorem 3.2 λ satisfies condition E, so that by Theorem 3.8, A is continuous on $C(S)_\beta$ into $C(T)_\beta$. To obtain the converse repeat this last argument.

Corollary 3.10: Any linear operator A with domain $C(S)$ and range $C(T)$ given by a kernel λ is continuous with the β -topology on both these spaces when S is paracompact.

Proof: For then A is weakly continuous into $C(T)$ with the topology of pointwise convergence on T .

To answer the question earlier posed on operator extensions from $C_0(S)$ to $C(S)$ we have

Theorem 3.11: Let A be a continuous linear operator on $C_0(S)$ into $C(T)_\beta$. Then A has an extension \bar{A} taking $C(S)$ into $C(T)$ provided the kernel of A satisfies condition E'. This

is then the only β or β' continuous extension of A to $C(S)$ and consequently, $\bar{A} = A^{**}|_{C(S)}$.

Proof: By Theorem 2.20, A has a kernel λ . If λ satisfies condition E' , then by Theorem 3.7 the formula $\bar{A}f = \lambda(f)$ defines a continuous linear operator on $C(S)_\beta$ into $C(T)_\beta$. Further for $f \in C_0(S)$, $\bar{A}f = \lambda(f) = Af$. The uniqueness of \bar{A} follows from Theorem 3.5 and moreover by Theorem 2.21 (4) for $f \in C(S)$ one has $A^{**}f = \lambda(f) = \bar{A}f$.

To complete our work in this section we prove

Theorem 3.12: Let A be a linear operator on $C(S)$ into $C(T)$. Then A is continuous from $C(S)_\beta$ into $(C(T), \|\cdot\|)$ if and only if A has a kernel representation λ satisfying the condition $\{\lambda(x): x \in T\}$ is β -equicontinuous.

Proof: If A is continuous on $C(S)_\beta$ into $(C(T), \|\cdot\|)$, then by Theorem 3.5 A has a kernel representation λ . But since it is clear that $\{x: x \in T\}$ is contained in the polar of the unit ball in $C(T)$ we have using Lemma 3.4 and Theorem 3.6 (2) that $\{\lambda(x): x \in T\}$ is β -equicontinuous.

Conversely suppose A has a kernel representation λ with the property $\{\lambda(x): x \in T\}$ is β -equicontinuous. This means there is a β -neighborhood U of 0 in $C(S)$ such that $|\int_S f(y) \lambda(x, dy)| \leq 1$ for all $f \in U$ and $x \in T$. But this says $U \subset A^{-1}(B)$ where B is the unit ball in $C(T)$ proving that A is continuous from $C(S)$ into $(C(T), \|\cdot\|)$.

This concludes our representation theory for β -continuous operators. The reader will notice that Theorem 3.12 remains

valid with β replaced by β' throughout. We will no longer deal with the β' topology; it is unwieldly and has served its purpose in that through its use we are able to obtain Theorem 3.8.

Before closing this section we note a few corollaries of our results and some open questions.

One of the more familiar linear operators given by a kernel λ is an operator of the form $[Af](x) = \int_T f(y)k(x,y)\mu(dy)$ where $k:TxS \rightarrow \mathbb{C}$, \mathbb{C} denoting the complex numbers and $\mu \in M(S)$ and $k(x,\cdot) \in L^1(S,\mu)$ for all $x \in T$ and $k(\cdot,y) \in C(T)$ for all $y \in S$. The kernel λ is of course given by $\lambda(x,E) = \int_E k(x,y)\mu(dy)$ for all Borel sets E and is a kernel provided $A(C_0(S)) \subset C(T)$. If for example T is metrizable and k is bounded this always holds, and in fact $A(B(S)) \subset C(T)$ by Lebesgue's Dominated Convergence Theorem. In general however, if S is paracompact (for example, if S is a topological group) and $A(C(S)) \subset C(T)$, then by Theorem 3.9 A is a continuous operator on $C(S)_\beta$ into $C(T)_\beta$ and given a compact set $K \subset T$ and an $\epsilon > 0$ there is a compact set $Q \subset S$ such that $\sup\{\int_{S \setminus Q} |k(x,y)| |\mu|(dy) : x \in K\} \leq \epsilon$ by Theorem 3.2 and Theorem 1.15. Further A is continuous on $C(S)_\beta$ into $(C(T), |||)$ if and only if there is such a set Q so that $\sup\{\int_{S \setminus Q} |k(x,y)| |\mu|(dy) : x \in T\} \leq \epsilon$ by Theorem 3.12. As we shall see in Chapter IV much more can be said along this line if $A(B(S)) \subset C(T)$.

Several open questions remain. For example in the light of Theorem 3.2 one asks the obvious question--If λ is a

continuous kernel does it follow that λ satisfies condition E'? A characterization of the β' -equicontinuous sets might yield the answer. Further it is not known whether β and β' are even distinct topologies when S is not paracompact. An answer to this question is equivalent to proving or disproving Theorem 3.8 with the β' topology on $C(T)$ rather than β ; one takes $\lambda(x) = \overset{0}{x}$ to see this.

Finally, the integral or kernel representation of operators on various function spaces has a long and involved history which touches many branches of both classical and modern analysis. See for example, Dunford and Schwartz [9] and Edwards [13], along with the work in various references listed therein and also the work found in Dynkin [11] as well as that of other probabilists.

Much of this work has been done with the aid of vector-valued measures. We believe our work is the most thorough of its type to date and that our work on operators on $C(S)_\beta$ is the first of its kind. We now use the results obtained here to study weakly compact and compact operators and their kernels, which will lead us to some interesting applications.

CHAPTER IV

WEAKLY CONTINUOUS KERNELS

It is in this chapter that our labor in non-metrizable locally compact spaces with arbitrary bounded Borel measures yields results of wide application, seemingly far removed from operator representation. To be more specific, let A be a bounded subset of $M(S)$ and let T be its weak* closure and topologize T with the weak* topology. By Aliagou's theorem, T is compact, and the natural mapping λ on T into $M(S)$ given by $\lambda(v, E) = v(E)$ for $v \in T$ and E a Borel set, defines a kernel on T , since $\lambda(f)(v) = \int_S f d v$ is continuous for all $f \in C_0(S)$ with the weak* topology on T ; its range is the weak* closure of A . With this technique we will apply our results, motivated by operator theory, to certain problems of weak and strong compactness in $M(S)$ and the L^1 spaces. We will supply rather brief proofs to certain known theorems due to various authors, as well as obtain some new results on weak convergence. These will be documented in the sequel.

Before beginning we would like to motivate the reader toward these results and to point out the specific known results on which they depend.

Suppose $\lambda: T \rightarrow M(S)$ is a kernel and suppose the operator $Af = \lambda(f)$ is a weakly compact operator on $C_0(S)$ into $C(T)_\beta$. By Theorem 1.8 and our Theorem 2.21, we have that $\lambda(\cdot, E) = A^{**}\psi_E \in C(T)$ for all Borel sets E . We will call such a

kernel a weakly continuous kernel and our remarks above establish the basic relationship between weakly compact operators and weakly continuous kernels which we will completely describe in the sequel.

Such kernels arise quite easily. Suppose that T is a metrizable space and that k is a real or complex valued function on $T \times S$ such that $k(x, \cdot) \in L^1(\mu)$ for all $x \in T$ and a fixed $\mu \in M(S)^+$. Define $\lambda: T \rightarrow M(S)$ by $\lambda(x, E) = \int_E k(x, y) \mu(dy)$. Since T is metrizable it follows from the dominated convergence theorem that $\lambda(\cdot, E)$ is continuous for all Borel sets E . We will prove a converse of this in the sequel for a σ -compact space T .

Our results hinge on the following theorems. We will make extensive use of two results, Theorems 1.15 and 1.16 due to Conway, along with Theorem 1.8 and our results, Theorems 2.21, 3.6 and 3.8 which essentially fall back to Theorem 2.19. Finally we will use Grothendieck's result on weak compactness in $M(S)$, Theorem 1.18; we note however that the portion of this theorem which will be of most use is the equivalence of (c) and (d) which Conway [6] obtains using Theorems 1.15 and 1.16 mentioned above, along with his generalization of Dieudonné's result, Theorem 1.22.

Our immediate goal is as follows. We wish to relate the continuity of a function $\lambda: T \rightarrow M(S)$ in the weak topology on $M(S)$ to the properties of its range as a subset of $M(S)$, and in turn relate these to its properties as an operator on

$C_0(S)$ into $C(T)$, and as an operator on $M(T)$ into $M(S)$ (see (3) of Theorem 2.21), and in turn relate all of these to the continuity of the functions $\lambda(\cdot, U)$ for open sets U contained in S .

We begin with the following.

Theorem 4.1: Let T be any topological space and $\lambda: T \rightarrow M(S)$ with $\|\lambda\| < \infty$. Then $\lambda(\cdot, U) \in C(T)$ for all open sets U if and only if $\lambda(f) \in C(T)$ for all bounded lower semicontinuous functions f on S .

Proof: Since the characteristic function of an open set is lower semicontinuous one implication is clear.

Conversely, suppose $\lambda(\cdot, U)$ is continuous for all open sets U . It suffices to show $\lambda(f) \in C(T)$ for any arbitrary lower semicontinuous function f such that $-1 < f(x) < 1$ for all $x \in S$.

Let $U_k^n = \{x \in S: k/n < f(x)\}$ for $k = 0, \pm 1, \dots, \pm n$. Since f is lower semicontinuous each set U_k^n is open and if $E_k^n = U_k^n \setminus U_{k+1}^n$ then $\lambda(\cdot, E_k^n) = \lambda(\cdot, U_k^n) - \lambda(\cdot, U_{k+1}^n)$ is continuous on T . For any $x \in T$ we then have

$$|\lambda(f)(x) - \sum_{k=-(n-1)}^{n-1} k/n \lambda(x, E_k^n)|$$

$$= \left| \int_S f(y) \lambda(x, dy) - \sum_{k=-(n-1)}^{n-1} k/n \lambda(x, E_k^n) \right|$$

$$= \left| \sum_{k=-(n-1)}^{n-1} \int_{E_k^n} (f(y) - k/n) \lambda(x, dy) \right|$$

$$\leq \sum_{k=-(n-1)}^{n-1} \int_{E_k^n} |f(y) - k/n| |\lambda|(x, dy)$$

$$\leq \sum_{k=-(n-1)}^{n-1} 1/n |\lambda|(x, E_k^n) \leq \|\lambda\|/n$$

so that $\lambda(f)$ is the uniform limit of the continuous functions

$$f_n(x) = \sum_{k=-(n-1)}^{n-1} k/n \lambda(x, E_k^n) \text{ and is therefore continuous.}$$

Corollary 4.2: Let $\{\mu_\alpha\}$ be a bounded net in $M(S)$ and $\mu \in M(S)$. Then $\mu_\alpha(U) \rightarrow \mu(U)$ for all open sets U if and only if $\int_S f d\mu_\alpha \rightarrow \int_S f d\mu$ for all bounded lower semicontinuous functions f on S .

Proof: Suppose $\mu_\alpha \rightarrow \mu$ on open sets U . Let $T = \{\mu\} \cup \{\mu_\alpha\}$ and topologize T as follows. If $\nu \in T$ and $\nu \neq \mu$ let $\{\nu\}$ and all sets containing $\{\nu\}$ be neighborhoods of $\{\nu\}$. Let μ have a base for neighborhoods consisting of all sets of the form $W(\mu, U, \epsilon) = \{\mu_\alpha \in T : |\mu_\alpha(U) - \mu(U)| < \epsilon\}$ for U an open set in S and $\epsilon > 0$. Define $\lambda: T \rightarrow M(S)$ by $\lambda(\nu, E) = \nu(E)$. With T topologized as above $\lambda(\cdot, U)$ is continuous on T for all open sets U and bounded since $\{\mu_\alpha\}$ is bounded. Hence by Theorem 4.1 $\lambda(f)$ is continuous on T for all bounded lower semicontinuous functions f on S . But $\mu_\alpha \rightarrow \mu$ in T so that $\lambda(f)(\mu_\alpha) = \int_S f d\mu_\alpha \rightarrow \int_S f d\mu = \lambda(f)(\mu)$.

A closed set $C \subset S$ is called a closed G_δ if there is a sequence $\{U_n\}$ of open sets containing C such that $C = \bigcap_{n=1}^{\infty} U_n$. It is easy to construct spaces in which not every closed set is a G_δ ; any non-first countable Hausdorff space will do. In a metric space every closed set is a G_δ .

Let C be a closed set whose complement is σ -compact. That is, $S \setminus C = \bigcup_{n=1}^{\infty} Q_n$ of a sequence of compact sets $\{Q_n\}$. Note S is Hausdorff and Q_n is closed and hence $C = \bigcap_{n=1}^{\infty} S \setminus Q_n$ is a G_δ . However, it is in general not true that every G_δ is a set of this form. We will call such a closed set C a strict G_δ .

Our next result, which is motivated by a theorem found in Edwards [13,p. 284], is a considerable strengthening of the known results on weak convergence and weak compactness in $M(S)$. This will be made clear in the sequel. We point out that our proof relies on the results of Conway mentioned above and that part of Grothendieck's theorem which can be obtained directly from these same results.

Theorem 4.3: Let T be a topological space and $\lambda: T \rightarrow M(S)$ such that $\|\lambda\| < \infty$ and $\lambda(\cdot, C)$ is continuous on T for each closed strict G_δ set C . Then λ is a kernel on T and for each compact set $K \subset T$ the set $\lambda(K) = \{\lambda(x): x \in K\}$ is weakly compact in $M(S)$.

Proof: We begin by showing that λ is a kernel on T . Let $f \in C_0(S)$, $0 \leq f \leq 1$. Let $A_k^n = \{x: f(x) \leq k/n\}$. Each set A_k^n is closed and $S \setminus A_k^n = \bigcup_{i=1}^{\infty} \{x: \frac{k}{n} + \frac{1}{i} \leq f(x)\}$ and so A_k^n is a strict G_δ for $k = 0, 1, \dots, n$. Now $A_k^n \subset A_{k+1}^n$. Let $E_k^n = A_{k+1}^n \setminus A_k^n = \{x: \frac{k}{n} < f(x) \leq \frac{k+1}{n}\}$ for $k = 0, 1, \dots, n-1$. Then $S = \bigcup_{k=0}^{n-1} E_k^n \cup A_0^n$ and $\lambda(\cdot, E_k^n) = \lambda(\cdot, A_{k+1}^n) - \lambda(\cdot, A_k^n) \in C(T)$. As in the proof of Theorem 4.1 we show that $\lambda(f)$ is the uniform limit of the sequence of continuous functions $\sum_{k=0}^{n-1} \frac{k}{n} \lambda(\cdot, E_k^n)$. Consequently λ is a kernel on T . Notice that this also means the mapping $x \rightarrow \lambda(x)$ of T into $M(S)$ is continuous with the weak* topology on $M(S)$.

Let K be a fixed compact set in T . We will show $\lambda(K)$ is weakly compact in $M(S)$. Since $x \rightarrow \lambda(x)$ is weak* continuous

it follows that $\lambda(K)$ is weak* compact and hence weak* closed and hence weakly closed. Consequently it is sufficient to show $\lambda(K)$ is weakly relatively compact in $M(S)$.

By Eberlein's theorem (Theorem 1.4) it suffices to show that every sequence $\{\lambda(x_n)\} \subset \lambda(K)$ has a weak cluster point in $M(S)$. But this will be true if it can be shown that the set $\{\lambda(x_n)\}$ is weakly relatively compact in $M(S)$. By Theorem 1.18(c) and Theorem 1.21 it suffices to show that the collection $\{\lambda(x_n)\}$ is uniformly inner regular with respect to open sets. This is what we will prove.

Since $\{x_n\} \subset K$ and K is compact then $\{x_n\}$ has a cluster point $x \in K$ and some subnet $\{x_\alpha\}$ of $\{x_n\}$ converges to x . Notice however that $\{x_\alpha\} \subset \{x_n\}$. We let $\mu_0 = \lambda(x)$, $\mu_n = \lambda(x_n)$ and $\mu_\alpha = \lambda(x_\alpha)$.

We will first show then $\mu_\alpha(U) \rightarrow \mu(U)$ for all open sets U .

Let U be a fixed open set. By the inner regularity of μ_0 there is a compact set $Q_1 \subset U$ such that $|\mu_0|(U \setminus Q_1) \leq 1$. Since S is locally compact and μ_1 is inner regular we can construct a compact set $Q_2 \subset U$ such that $Q_1 \subset Q_2^\circ$, where Q_2° denotes the interior of Q_2 , and $|\mu_k|(U \setminus Q_2) \leq 1/2$ for $k=0,1$. By induction we can construct a sequence $\{Q_n\}$ of compact subsets of U such that $Q_n \subset Q_{n+1}^\circ$ for all n and $|\mu_k|(U \setminus Q_n) \leq 1/n$ for $k = 0,1,\dots,n-1$. Let $V = \bigcup_{k=0}^{\infty} Q_k = \bigcup_{k=0}^{\infty} Q_k^\circ$ and let $C = S \setminus V$. Since V is σ -compact the set C is a strict G_δ and consequently

$\lambda(\cdot, V) = \lambda(\cdot, S) - \lambda(\cdot, C)$ is continuous on T . We claim $|\mu_k|(U \setminus V) = 0$ for $k = 0, 1, \dots$. For, $|\mu_k|(U \setminus V) \leq |\mu_k|(U \setminus Q_n) \leq 1/n$ for all $n \geq k+1$. Hence given an open set U we can find an open σ -compact set V such that $|\mu_n|(U \setminus V) = 0$ for all n and consequently $\mu_n(U) = \mu_n(V)$ for all n .

Since $x_\alpha \rightarrow x$ and $\lambda(\cdot, V)$ is continuous then $\lambda(x_\alpha, V) \rightarrow \lambda(x, V)$. But $\{x_\alpha\} \subset \{x_n\}$ and for each α , $\lambda(x_\alpha, V) = \lambda(x_n, V) = \mu_n(V) = \mu_n(U) = \mu_\alpha(U)$ for some n and similarly, $\lambda(x, V) = \mu_0(U)$. Hence $\mu_\alpha(U) \rightarrow \mu_0(U)$. Consequently by Corollary 4.2, $\int_S f d\mu_\alpha \rightarrow \int_S f d\mu_0$ for all bounded lower semicontinuous functions f on S .

We now show $\{\mu_n\}$ is uniformly inner regular with respect to open sets. Again let U be a fixed open set and construct the open σ -compact set V as before so that $|\mu_n|(U \setminus V) = 0$ for $n = 0, 1, \dots$.

Let ν_n be the restriction of μ_n to V so that $\nu_n \in M(V)$ for all n . To show that $\{\mu_n\}$ is uniformly inner regular it suffices, by Theorem 1.15, to show that $\{\nu_n\}$ is β -equicontinuous as a subset of the dual of $C(V)_\beta$. Since V is σ -compact and hence paracompact, it suffices to show that $\{\nu_n\}$ is β -weak* countably compact and to do this it suffices to show that $\{\nu_n\}$ has a β -weak* cluster point in $M(V)$.

Let $g \in C(V)$ and suppose $g \geq 0$. Let $f(x) = g(x)$ for all $x \in V$ and $f(x) = 0$ for $x \notin V$. Since V is open, f is a bounded lower semicontinuous function on S . Consequently

$\int_S f d\mu_\alpha \rightarrow \int_S f d\mu_0$. But $\int_V g d\nu_n = \int_S f d\mu_n$ for all $n = 0, 1, \dots$

and consequently a subnet of $\{\int_V g d\nu_n\}$ converges to $\int_V g d\nu_0$ and therefore ν_0 is a β -weak* cluster point of $\{\nu_n\}$.

Hence $\{\nu_n\}$ is β -equicontinuous in $M(V)$ and so there is a compact set $Q \subset V$, and hence compact in S , such that

$$|\mu_n|(U \setminus Q) = |\mu_n|(V \setminus Q) = |\nu_n|(V \setminus Q) < \varepsilon \text{ for a given } \varepsilon > 0 \text{ or,} \\ |\lambda|(x_n, U \setminus Q) < \varepsilon \text{ for all } n = 1, 2, \dots, \text{ completing the proof.}$$

We now prove as a corollary a strengthening of the result found in Edwards [13, p.284] which illustrates the applicability of our result to weak compactness in the space of measures.

Let α denote the weak topology on $M(S)$ generated by the collection of all characteristic functions of closed strict G_δ subsets of S . That is, α is the topology on $M(S)$ whose base of neighborhoods at a point $\mu \in M(S)$ consists of all sets $U(\mu, C, \varepsilon) = \{\nu \in M(S) : |\mu(C) - \nu(C)| < \varepsilon\}$ for all $\varepsilon > 0$ and closed strict G_δ sets C . By Corollary 4.2, α is a stronger topology on $M(S)$ than the weak* topology and is therefore Hausdorff.

Corollary 4.4: Let A be a bounded subset of $M(S)$. Then A is weakly relatively compact in $M(S)$ if and only if A is relatively compact for the α topology on $M(S)$.

Proof: We let B be the weak closure of A and C the α -closure of A . We show that B is weakly compact if and only if C is α -compact. Suppose C is α -compact and let T denote the set C with the α -topology. Let $\lambda: T \rightarrow M(S)$ be defined by $\lambda(\nu, E) = \nu(E)$ for $\nu \in C$ and E Borel. Then $\lambda(\cdot, C)$ is continuous on T for all closed strict G_δ sets C and so since T is compact,

$\lambda(T) \supset B$ is weakly compact by Theorem 4.3 and consequently B is weakly compact. Conversely suppose A is weakly relatively compact. Since the weak topology is stronger than the α topology then B being weakly compact is then α compact and hence α closed. Consequently $C = B$ is α -compact completing the proof.

We come now to the principle result of this chapter which relates all the properties of a kernel λ considered as an operator, or a vector valued function, or as simply defining a subset of $M(S)$, with respect to the idea of weak compactness.

Theorem 4.5: Let $\lambda: T \rightarrow M(S)$ be a kernel, where T is a locally compact Hausdorff space. The following statements are equivalent.

- (1) The function $x \rightarrow \lambda(x)$ is weakly continuous.
- (2) For each compact subset $K \subset T$ the set $\lambda(K) = \{\lambda(x) : x \in K\}$ is weakly compact in $M(S)$.
- (3) If $\{f_n\} \subset C_0(S)$ and $f_n \rightarrow 0$ weakly, then $\lambda(f_n) \rightarrow 0$ uniformly on compact subsets of T .
- (4) If $\{U_n\}$ is a collection of open mutually disjoint subsets of S , then $\lambda(\cdot, U_n) \rightarrow 0$ uniformly on compact subsets of T .
- (5) The set $\lambda(K)$ is uniformly inner regular with respect to open sets for each compact subset K of T .
- (6) The set $\lambda(K)$ is β -equicontinuous and uniformly outer regular with respect to compact sets for

each compact subset K of T .

- (7) The transformation $\mu \rightarrow \lambda(\mu)$ where $\lambda(\mu)(E) = \int_T \lambda(x, E) \mu(dx)$ takes β -equicontinuous sets in $M(T)$ into weakly relatively compact sets in $M(S)$.
- (8) The set $\{\lambda(f) : f \in C(S), \|f\| \leq 1\}$ is weakly relatively compact in $C(T)_\beta$.
- (9) The set $\{\lambda(f) : f \in C_0(S), \|f\| \leq 1\}$ is weakly relatively compact in $C(T)_\beta$.
- (10) For all $f \in \bar{B}(S)$, $\lambda(f) \in C(T)$.
- (11) $\lambda(\cdot, E) \in C(T)$ for all $E \in \Sigma$ where Σ is any one of the following:
 - (a) Σ consists of all Borel sets.
 - (b) Σ consists of all open sets.
- (12) $\lambda(\cdot, C) \in C(T)$ for all closed strict G_δ sets C .

Proof: The equivalences of (2) through (6) follow immediately upon application of Grothendieck's theorem, Theorem 1.18. Clearly (1) implies (2) since a continuous function maps compact sets onto compact sets. We show that (2) implies (1). Let M denote the set $\lambda(K)$ with the weak topology and N this same set with the weak* topology. By (2), M is compact and since N is Hausdorff the identity map of M onto N is a homeomorphism and thus the weak and weak* topologies agree on $\lambda(K)$. Let $x_\alpha \rightarrow x$ in T and let U be a neighborhood of x with compact closure and $x_\alpha \in \bar{U}$ for $\alpha \geq \alpha_0$. Then $\lambda(x) \in \lambda(\bar{U})$ and $\{\lambda(x_\alpha)\} \subset \lambda(\bar{U})$ for $\alpha \geq \alpha_0$. Since λ is a kernel,

$\lambda(x_\alpha) \rightarrow \lambda(x)$ weak* for $\alpha \geq \alpha_0$ and since \bar{U} is compact, the weak and weak* topologies agree on $\lambda(\bar{U})$ completing the proof of (1).

Hence (1) through (6) are equivalent. We show that (5) implies (7). Let H be a β -equicontinuous set in $M(T)$. Then $\alpha = \sup\{\|\mu\| : \mu \in H\} < \infty$ and if $\varepsilon > 0$ there is a compact set $K \subset T$ such that $|\mu|(T \setminus K) < \varepsilon/2\|\lambda\|$ for all $\mu \in H$ by Theorem 1.15. By (5) given an open set $U \subset S$ there is a compact set $Q \subset U$ such that $|\lambda|(x, U \setminus Q) < \varepsilon/2\alpha$ for all $x \in K$. It now easily follows that $|\lambda(\mu)|(U \setminus Q) < \varepsilon/2\alpha |\mu|(K) + \|\mu\| \varepsilon/2\|\lambda\| < \varepsilon$ for all $\mu \in H$, showing that $\lambda(H)$ is uniformly inner regular with respect to open sets and hence weakly relatively compact.

Supposing that (7) holds let us prove (8). Let K be a compact subset of T and $H = \{x : x \in K\}$. Then $\lambda(H)$ is weakly relatively compact in $M(S)$ since H is β -equicontinuous in $M(T)$. In particular $\lambda(H)$ is β -equicontinuous and since λ is a kernel we apply Theorem 3.8 to see that the formula $Af = \lambda(f)$ for $f \in C(S)$ defines a continuous linear operator on $C(S)_\beta$ into $C(T)_\beta$ with adjoint given by $A^*\mu = \lambda(\mu)$ by Theorem 3.6. It now follows from this and (7) along with Theorem 1.8 that A takes β -bounded and hence norm bounded subsets of $C(S)$ into weakly relatively compact subsets of $C(T)_\beta$ proving (8).

Clearly (8) implies (9). Assuming (9) is valid set $Af = \lambda(f)$ for $f \in C_0(S)$. From (9) A is a weakly compact operator on $C_0(S)$ in $C(T)_\beta$ and hence by Theorem 1.8 $A^{**}(C_0(S)^{**}) \subset C(T)$.

In particular $\lambda(f) = A \ast f \in C(T)$ for all $f \in B(S)$ by Theorem 2.21 proving (10).

Clearly (10) implies 11(a) and 11(a) implies 11(b) and 11(b) implies (12). But by Theorem 4.3 (12) implies (2) completing the proof.

We remark that the overriding hypothesis " λ is a kernel" is redundant for certain of the various implications, in particular parts (8) through (12), while local compactness of T is not essential for the equivalences (2) through (6).

Consequently a weakly continuous kernel satisfies all the equivalent conditions of Theorem 4.5 and we point out the following corollary of this and Theorem 4.3.

Corollary 4.6: If T is a locally compact Hausdorff space and $\lambda: T \rightarrow M(S)$ with $\|\lambda\| < \infty$, then λ is a weakly continuous kernel if and only if $\lambda(\cdot, C) \in C(T)$ for all closed strict G_δ sets C .

Let us pause and consider a question related to our work above and that of Dieudonné [7] and Conway [6] on quarrable sets. Given a kernel λ , does it define a class of subsets Σ of S such that continuity of $\lambda(\cdot, E)$ for all $E \in \Sigma$ implies continuity of $\lambda(\cdot, E)$ for all Borel sets E ? Further is the class of closed strict G_δ sets the minimal class having this property for all λ ?

An important virtue of our work thus far is that it has not been restricted to kernels whose range lies in $M(S)^+$.

We now consider certain questions involving the mapping $x \rightarrow |\lambda|(x)$ for a given map $\lambda: T \rightarrow M(S)$. Unless otherwise noted, T is locally compact and Hausdorff and we suppose that all measures $\lambda(x)$ are real signed measures.

Let us begin with an example which illustrates one of the difficulties in relating λ to $|\lambda|$. Let T be the interval $[0,1]$ and let S be a two point space $\{a,b\}$ with the discrete topology. For $x \neq 0$, let $\lambda(x,\{a\}) = 1$, $\lambda(x,\{b\}) = -1$, so that $\lambda(x,S) = 0$, $|\lambda|(x,S) = 2$. Let $\lambda(0,\{a\}) = \lambda(0,\{b\}) = 1$. Then $|\lambda|(\cdot, U)$ is continuous for all open sets U and hence $|\lambda|$ is a weakly continuous kernel; but λ is not even a kernel. For if $x \neq 0$ then $\lambda(f)(x) = f(a) - f(b)$ while $\lambda(f)(0) = f(a) + f(b)$. Consequently one can have $|\lambda|$ a weakly continuous kernel even though λ is not a kernel.

Conversely, one may ask whether one can have λ a weakly continuous kernel and $|\lambda|$ not even a kernel? The answer again is the same, and we proceed to construct an example. In essence, we will construct a sequence of measures of constant non-zero norm which is weakly convergent to 0.

Again let $T = [0,1]$ and let $S = (0,2\pi)$. For E any closed, open or half open interval in S with endpoints $a < b$, set $\lambda(x,E) = x(\sin \frac{b}{x} - \sin \frac{a}{x})$ for $x \neq 0$ and $\lambda(0,E) = 0$. It is easily seen that $\lambda(x,E) = \int_E \cos \frac{t}{x} dt$ for $x \neq 0$ and all Borel sets E , where the integral is taken with respect to Lebesgue measure on $(0,2\pi)$.

If U is an open set in S , then by a well-known theorem $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$ of mutually disjoint open intervals, $(a_n, b_n) \subset S$. Let $g_n(x) = x(\sin \frac{b_n}{x} - \sin \frac{a_n}{x}) = \lambda(x, (a_n, b_n))$ for $x \neq 0$ and $g(0) = 0$. Then g_n is continuous on T and $\lambda(x, U) = \sum_{n=1}^{\infty} g_n(x)$ for all $x \in T$. We show then the convergence is uniform on T . By the mean value theorem, $|g_n(x)| = |\cos \frac{c_n(x)}{x} (b_n - a_n)| \leq |b_n - a_n|$ for all $x \neq 0$ where $c_n(x) \in (a_n, b_n)$. But $\sum_{n=1}^{\infty} |b_n - a_n| \leq 2\pi$, so that $\sum g_n(x)$ converges uniformly making

$\lambda(\cdot, U)$ continuous on T . By Theorem 4.1, λ is a kernel provided we can show that $\|\lambda\| < \infty$.

Since the function $y \rightarrow x \sin y/x$ has amplitude x and period $2\pi x$ it then has total variation $2x(2\pi/2\pi x) = 2$ so that $|\lambda|(x, S) = 2$ for all $x \neq 0$ while $|\lambda|(0, S) = 0$. Hence $\|\lambda\| < \infty$ and moreover $|\lambda|(\cdot, S)$ is not continuous on T .

By Theorem 4.5 (11), λ is a weakly continuous kernel but $|\lambda|$ is not a kernel.

The next few results clarify this matter completely. We point out in passing that the above examples indicate that improving the spaces S and T does not lead to improved results.

Our first example above shows that the hypothesis of the next corollary cannot be weakened.

Corollary 4.7: Let $\lambda: T \rightarrow M(S)$ be a kernel such that $|\lambda|$ is a weakly continuous kernel. Then λ is a weakly continuous kernel.

Proof: Since $|\lambda|$ is a weakly continuous kernel then $|\lambda|$ satisfies condition (5) of Theorem 4.5 and consequently so does λ . But since λ is a kernel this means λ satisfies all the conditions of Theorem 4.5 and so is a weakly continuous kernel.

Theorem 4.8: Let $\lambda:T \rightarrow M(S)$ be real-valued and $\|\lambda\| < \infty$. The following are equivalent.

- (1) λ and $|\lambda|$ are weakly continuous kernels.
- (2) λ is a kernel and $|\lambda|(\cdot, C)$ is continuous for all closed strict G_δ sets C .
- (3) λ is a weakly continuous kernel and $|\lambda|(\cdot, S)$ is continuous.
- (4) $\lambda(\cdot, C)$ is continuous for all closed strict G_δ sets C , and $|\lambda|(\cdot, S)$ is continuous.

Proof: Clearly (1) implies (2) using Theorem 4.5. Since S is a closed strict G_δ then using Corollary 4.7 we see that (2) implies (3). Furthermore Theorem 4.5 implies that (3) implies (4). We have only to show that (4) implies (1).

If (4) holds, then λ is a kernel by Theorem 4.3 and so is a weakly continuous kernel by Theorem 4.5, and therefore $\lambda(\cdot, E)$ is continuous for all Borel sets E . Hence if C is a closed set then $|\lambda|(x, C) = \sup \left\{ \sum_{i=1}^n |\lambda(x, E_i)| : \{E_i\} \text{ is a partition by Borel sets of } C \right\}$ is lower semicontinuous. Since $|\lambda|(\cdot, S)$ is continuous, if U is an open set in S , then $|\lambda|(x, U) = |\lambda|(x, S) - |\lambda|(x, S \setminus U)$ is then upper semicontinuous.

But since λ is a kernel then $|\lambda|(\cdot, U)$ is also lower semicontinuous by Corollary 2.9 and hence $|\lambda|(\cdot, U)$ is continuous for all open sets U . By Theorem 4.1 and (11) of Theorem 4.5, $|\lambda|$ is a weakly continuous kernel. Clearly λ is a weakly continuous kernel completing the proof of (1) and of the theorem.

Corollary 4.9: If $\lambda: T \rightarrow M(S)$ is real-valued and satisfies any one of the conditions of Theorem 4.8, then λ^+ and λ^- are weakly continuous kernels.

We pause for a moment to examine certain overriding hypotheses in the above theorems. For example we usually suppose $\|\lambda\| < \infty$. This is always true if $x \rightarrow \lambda(x)$ is weak $*$ bounded and continuous; that is, if λ is a kernel. Dieudonné [7] has shown that the boundedness of a sequence of measures $\{\mu_n\}$ follows from convergence on certain classes of Borel sets. The relation between functions λ for which $\lambda(f)$ is continuous for all functions $f \in C_0(S)$ and those for which $\lambda(\cdot, U)$ is continuous for all open sets U , is completely unresolved in the absence of the condition $\|\lambda\| < \infty$. If one is given that $\|\lambda\| < \infty$, we have given some sufficient conditions --see Theorem 4.1--in order that λ be a kernel. Certain of the conditions in Theorem 4.5 yield the same conclusion, namely (1), and (8) through (12), as noted earlier.

Our next goal is to show how our results above can be made to yield both improvements and generalizations of certain

known results about weak convergence in $M(S)$. A generalization of Theorem 1.22 illustrates this.

Theorem 4.10: Let $\{\mu_\alpha\}$ be a bounded net in $M(S)$ and let $\mu \in M(S)$. The following are equivalent.

- (1) $\mu_\alpha \rightarrow \mu$ weakly and $\{\mu_\alpha\}$ is weakly relatively compact.
- (2) $\mu_\alpha \rightarrow \mu$ and $\{\mu_\alpha\}$ is relatively compact for the weak topology on $M(S)$ defined by the collection of all characteristic functions of closed strict G_δ sets.
- (3) (a) $\mu_\alpha \rightarrow \mu$ weak*
 (b) $\{\mu_\alpha\}$ is weakly relatively compact; i.e., $\{\mu_\alpha\}$ is β -equicontinuous and uniformly outer regular with respect to compacta.

Proof: Clearly (1) implies (2) by Corollary 4.4 and the fact that $\chi_C \in M(S)^*$ for all closed strict G_δ sets C . Again using Corollary 4.4 along with (c) of Theorem 1.18, we see that (2) implies 3(b), and by Theorem 4.3, (2) implies 3(a) the proof being exactly the same as that of Corollary 4.2. It remains to show that (3) implies (1). By 3(b) the weak closure T of $\{\mu_\alpha\}$ is weakly compact and hence weak* compact and therefore weak* closed. By 3(a) this means $\mu \in T$ and giving T the weak* topology we have that T is compact and $\mu_\alpha \rightarrow \mu$ in T . Define $\lambda: T \rightarrow M(S)$ by $\lambda(\nu, E) = \nu(E)$. With the weak* topology on T λ is a kernel and $\lambda(T) = T$ is weakly compact as a subset of $M(S)$ and therefore λ satisfies (2) of Theorem 4.5 and consequently also satisfies (1). But since $\mu_\alpha \rightarrow \mu$ in T ,

(1) says that $\mu_\alpha = \lambda(\mu_\alpha) \rightarrow \mu = \lambda(\mu)$ weakly. By our remarks above on the weak compactness of T this concludes the proof of (1).

It is easy to see that Theorem 4.10 is a generalization and improvement of Theorem 1.22 and we state this as

Corollary 4.11: Let $\mu \in M(S)$, $\{\mu_n\}$ a bounded sequence in $M(S)$. These are equivalent.

(1) $\mu_n \rightarrow \mu$ weakly.

(2) $\mu_n(C) \rightarrow \mu(C)$ for every closed strict G_δ set C .

(3) (a) $\mu_n \rightarrow \mu$ weak*.

(b) $\{\mu_n\}$ is β -equicontinuous and uniformly outer regular with respect to compacta.

A proof of Corollary 4.11 independent of Theorem 4.10 may be given as follows. Given a sequence $\{\mu_n\} \subset M(S)$ and $\mu \in M(S)$, let T be the one point compactification of the integers with ω denoting the point at infinity. Define $\lambda: T \rightarrow M(S)$ by $\lambda(n, E) = \mu_n(E)$ and $\lambda(\omega, E) = \mu(E)$. Then λ is a kernel if and only if $\mu_n \rightarrow \mu$ weak* and it is easily seen how to apply this to a proof of Corollary 4.11 or to gain other information about sequential convergence in $M(S)$ through the use of Theorem 4.5.

We now give an application to weak compactness in a space $L^1(\mu)$ essentially like the Dunford-Pettis theorem (see Edwards [13, p.274]). There is a natural mapping ϕ from $L^1(\mu)$ into $M(S)$, where μ is a non-negative regular Borel measure

on S , given by $\phi(f)(E) = \int_E f d\mu$ for $f \in L^1(\mu)$ and E a Borel set. Further, it is known that $\|\phi(f)\| = \int_S |f| d\mu = \|f\|_1$ (see [13]), so that ϕ is an isometry. The adjoint of $L^1(\mu)$ is of course the space L_∞ of all μ -essentially bounded measurable functions f with the essential supremum norm, $\|f\|_\infty = \inf\{r: |f(x)| \leq r \text{ almost everywhere } \mu\}$, and there is a natural mapping of $B(S)$ into L_∞ which is norm decreasing.

The weak topology on L^1 , $\sigma(L^1, L_\infty)$, is the topology with neighborhood base at $f \in L^1$ consisting of all sets $U(f, h, \epsilon) = \{g \in L^1: |\int_S (f-g)h d\mu| < \epsilon\}$ for $\epsilon > 0$ and a fixed $h \in L_\infty$. We define the α_1 topology on L^1 to be the topology with neighborhood base all sets $U(f, \chi_C, \epsilon)$, where C is a closed strict G_δ set. Note that $\chi_C \in L_\infty$ and the α_1 topology is weaker than $\sigma(L^1, L_\infty)$.

With this in mind we prove

Theorem 4.12: Let N be a bounded subset of $L^1(\mu)$. The following are equivalent.

- (1) N is $\sigma(L^1, L_\infty)$ compact.
- (2) N is α_1 compact.
- (3) N is weakly closed and for each open subset U of S and each $\epsilon > 0$ there is a compact set $K \subset U$ such that $\int_{U \setminus K} |f| d\mu < \epsilon$ for all $f \in N$.

Proof: Clearly (1) implies (2) by our remarks above. If (2) holds, we let T denote the set N with the α_1 topology and define $\lambda: T \rightarrow M(S)$ by $\lambda(f, E) = \int_E f d\mu$. If C is a closed strict G_δ set then $\lambda(f, C) = \int_C f d\mu$ so that $\lambda(\cdot, C)$ is continuous.

Further $\|\lambda(f)\| = \int_S |f| d\mu = \|f\|_1$ and therefore $\|\lambda\| < \infty$ since N is bounded. By Theorem 4.3 λ is a kernel and λ satisfies (2) of Theorem 4.5 and so also satisfies (5) of the same theorem. Since T is compact, this means that given an open set U and an $\varepsilon > 0$ there is a compact set $K \subset U$ such that $|\lambda(f)|(U \setminus K) = \int_{U \setminus K} |f| d\mu < \varepsilon$ for all $f \in T = N$. It remains to show that N is weakly closed. We have just shown that the image of N under λ , namely $\lambda(T)$, is weakly compact in $M(S)$ and hence weakly closed. The natural map ϕ being an isometry is continuous and hence weakly continuous. Thus $N = \phi^{-1}\lambda(T)$ is weakly closed since ϕ is 1-1. Consequently (3) holds.

We complete the proof by showing that (3) implies (1). We note first that $\phi(N)$ is weakly compact by Theorems 1.18 and 1.21 and hence weak* compact. Define λ on the set $T = \phi(N)$ with the weak* topology by $\lambda(\phi(f), E) = \int_E f d\mu = \phi(f)(E)$ for all Borel sets E . Clearly λ is well-defined and is a kernel on T with the weak* topology on that space and T is compact as noted above. Now by (3) λ satisfies (5) of Theorem 4.5 and so also satisfies (10) of the same theorem. To show that N is $\sigma(L^1, L_\infty)$ compact, let $\{f_\alpha\}$ be a net in N . Then $\{\phi(f_\alpha)\}$ is a net in the weak* compact set $\phi(N)$ and so has a cluster point $\phi(f) \in T = \phi(N)$.

Let $h \in L_\infty$ and define $g \in B(S)$ by $g(x) = h(x)$ if h is defined and $h(x) \leq \|h\|_\infty$ and $g(x) = 0$ otherwise. Then h and g differ at most on a set of μ measure 0. By (10) of Theorem 4.5 $\lambda(g)$ is continuous on T so that $\{\lambda(g)(\phi(f_\alpha))\}$ clusters to $\lambda(g)(\phi(f))$

But this says $\{\int_S g f_\alpha d\mu\}$ clusters to $\int_S g f d\mu$. Since $\int_S h f_\alpha d\mu = \int_S g f_\alpha d\mu$ for all α , and similarly for f , we have proven that $\{f_\alpha\}$ clusters to $f \in N$ in the topology $\sigma(L^1, L_\infty)$ completing the proof of (1).

This theorem completes our applications of kernel continuity to weak compactness and convergence in $M(S)$ and $L^1(\mu)$. Although we have not nearly exhausted the various possibilities and corollaries we feel that the methods available for applying our results should be clear.

We now turn to a brief description of kernel continuity in yet another topology, namely the strong or norm topology on $M(S)$.

Theorem 4.13: Let $\lambda: T \rightarrow M(S)$ be a kernel. The following are equivalent.

- (1) The function $x \rightarrow \lambda(x)$ is continuous in the norm topology on $M(S)$.
- (2) For each compact set $K \subset T$ the set $\lambda(K)$ is norm compact.
- (3) $\{\lambda(f) : f \in C(S), \|f\| \leq 1\}$ is β -relatively compact in $C(T)$.
- (4) $\{\lambda(f) : f \in C_0(S), \|f\| \leq 1\}$ is β -relatively compact in $C(T)$.
- (5) The mapping $\mu \rightarrow \lambda(\mu)$, where $\lambda(\mu)(E) = \int_T \lambda(x, E) \mu(dx)$, takes β -equicontinuous subsets of $M(T)$ into relatively compact subsets of $M(S)$.

Proof: Clearly (1) implies (2). To see that (2) implies (3) let $B = \{f \in C(S) : \|f\| \leq 1\}$ and let $F = \lambda(B)$. Then $\{\lambda(f)(x) : f \in B\} = \{g(x) : g \in F\}$ is bounded by $\|\lambda\|$. By Ascoli's theorem (Theorem 1.1) F will be compact-open compact provided F is equicontinuous. Let $x_0 \in T$ and let U be a neighborhood of x_0 with compact closure K . By (2), $\lambda(K)$ is norm compact in $M(S)$ and the identity map from $\lambda(K)$ with the norm topology onto $\lambda(K)$ with the weak* topology is continuous and hence a homeomorphism. Since λ is a kernel and therefore $x \mapsto \lambda(x)$ is weak* continuous then $\lambda(K)$ is also weak* closed so that the norm and weak* topologies agree on $\lambda(K)$ and since $x \mapsto \lambda(x)$ is weak* continuous on $U \subset K$ this means $x \mapsto \lambda(x)$ is norm continuous on U . Hence given $\epsilon > 0$ there is a neighborhood V of x_0 , $V \subset U$ such that $x \in V$ implies $\|\lambda(x) - \lambda(x_0)\| < \epsilon$ and therefore for all $f \in B$ $|\lambda(f)(x) - \lambda(f)(x_0)| \leq \|f\| \|\lambda(x) - \lambda(x_0)\| < \epsilon$ for all $x \in V$ proving that F is equicontinuous and being uniformly bounded is compact-open and hence β -relatively compact proving (3).

Clearly (3) implies (4). To see that (4) implies (5) let A be the continuous linear operator on $C_0(S)$ into $C(T)_\beta$ defined by $Af = \lambda(f)$. By (4) A takes the unit ball in $C_0(S)$ into a relatively compact set in $C(T)_\beta$ and so by Theorem 1.9, A^* takes β -equicontinuous sets into relatively compact subsets of $M(S)$. But by Theorem 2.21 $A^*\mu = \lambda(\mu)$ completing the proof of (5).

Clearly (5) implies (2) for the set $\{\overset{O}{x}: x \in K\}$ is β -equicontinuous for K a compact subset of T and therefore $\lambda(K)$ is relatively compact but also weak* closed and hence norm closed. Thus (2) holds and repeating the arguments involved in the proof of (2) implies (3) yields (1), completing the proof of the theorem.

This completes our work in this chapter. We will now apply the results obtained here to the representation and characterization of the compact and weakly compact operators on $C(S)_\beta$ along with further results on the general weakly continuous kernel.

We feel that the above material constitutes a useful bridge between operator theory and various notions of compactness and convergence in the space of measures and provides a tool of wide applicability to related problems. Perhaps the most interesting open question related to this is the determination of conditions on the mapping $\lambda: T \rightarrow M(S)$ so that $\|\lambda\| < \infty$ in terms of continuity of $\lambda(\cdot, E)$ over certain classes of sets E as has been done by Dieudonné [7] for sequences of measures.

There is in general a relation between the topology of a space and the kernels it can admit. For example, if T is a discrete space then any kernel on T satisfies the conditions of any of the theorems of this chapter; in particular, if $S = T$ and T is discrete, then $\lambda(x) = \overset{O}{x}$ is a kernel on T and (9) of Theorem 4.5 says the unit ball in $C(S)_\beta$ is weakly

compact and from (3) of Theorem 4.13, the stronger conclusion that the unit ball is β -compact also follows. A natural question is whether the topology of a space can be characterized by the kernels it admits in certain spaces of measures; of course, this is very closely related to the continuous operators it will admit on its space of continuous functions.

There remain certain intrinsic questions for a given kernel. Given a kernel λ , does it define a class of Borel sets Σ such that the continuity of $\lambda(\cdot, E)$ for $E \in \Sigma$ implies the conclusions of say Theorem 4.5? Also one may attempt to define new topologies on S through a kernel λ . See for example the work of Dynkin [11] on the intrinsic topology of a Markov process.

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CHAPTER V
COMPACT AND WEAKLY COMPACT OPERATORS
ON $C_0(S)$ AND $C(S)_\beta$

The linear topological space $C(S)_\beta$ has many pleasing properties. Its dual is a well-known Banach space and its bounded sets are precisely the norm bounded sets. Further properties are stated in Theorem 1.14. Unfortunately however, $C(S)_\beta$ is neither barrelled nor bornological nor metrizable unless S is compact and in fact any one of these is equivalent to the compactness of S (see Conway [6]). For this reason much of the general theory of linear spaces, and especially operator theory, does not apply to $C(S)_\beta$. In this chapter we will characterize the compact and weakly compact operators on $C(S)_\beta$ with the aid of kernel functions. We will see that here again the β -topology on $C(S)$ is a very natural one to use and that it is more manageable than the norm topology when S is not compact.

An operator on a Banach space X into a Banach space Y is called compact (weakly compact) if it maps bounded sets in X into relatively compact (weakly relatively compact) subsets of Y . But in a Banach space every bounded absolutely convex absorbent set is also a neighborhood of the zero vector so that the operator also maps neighborhoods of zero into relatively compact sets. In making analogous definitions for linear spaces one has the choice of using neighborhoods of

zero or bounded sets. Much of the general theory (see Edwards [13, pp. 616-674] or [28, pp. 142-154]) yields results for operators taking bounded sets into relatively compact sets. These results are analogous to the results known for Banach spaces.

We will call an operator A on $C(S)_\beta$ a compact (weakly compact) operator if A maps the unit ball in $C(S)$ into a relatively compact (weakly relatively compact) set. The operator A will be called β -compact (β -weakly compact) if there is a β -neighborhood V of 0 such that $A(V)$ is relatively compact (weakly relatively compact).

We consider the weakly compact case first and, as is now usual, begin with a result about kernels. In the sequel, T denotes a locally compact Hausdorff space, as does S .

Theorem 5.1: Let $\lambda: T \rightarrow M(S)$ be a kernel. There is a β -neighborhood V of zero such that $\lambda(V) = \{\lambda(f): f \in V\}$ is weakly relatively compact in $C(T)_\beta$ if and only if $\lambda(T) = \{\lambda(x): x \in T\}$ is β -equicontinuous and $\lambda(K) = \{\lambda(x): x \in K\}$ is weakly compact in $M(S)$ for every compact subset K of T .

Proof: Suppose there is a β -neighborhood V of zero in $C(S)$ such that $\lambda(V)$ is weakly relatively compact in $C(T)_\beta$. In particular since neighborhoods are absorbent this means $\lambda(f) \in C(T)$ for all $f \in C(S)$ and so the formula $Af = \lambda(f)$ defines a linear operator A on $C(S)$ into $C(T)$ such that $A(V)$ is weakly relatively compact in $C(T)_\beta$ and consequently β -bounded,

and hence norm bounded by Theorem 1.14(b). Hence A is continuous from $C(S)_\beta$ into $(C(T), \|\cdot\|)$ and by Theorem 3.12 $\lambda(T)$ is β -equicontinuous. Furthermore since β -neighborhoods absorb bounded sets in $C(S)$ we also have that $\lambda(B)$ is weakly relatively compact in $C(T)_\beta$ where B is the unit ball in $C(S)$. Hence λ satisfies (8) of Theorem 4.5 and so also (2) of the same theorem.

Conversely, suppose $\lambda(K)$ is weakly compact in $M(S)$ for each compact subset K of T and that $\lambda(T)$ is β -equicontinuous. Then by Theorem 3.1 $\lambda(f) \in C(T)$ for each $f \in C(S)$ since λ satisfies condition E and furthermore there is a function $\phi \in C_0(S)$ $\phi \geq 0$ such that $\|\frac{1}{\phi} \cdot \lambda(x)\| \leq 1$ for all $x \in T$ by the β -equicontinuity of $\lambda(T)$ and Theorem 1.15(c), where $[\frac{1}{\phi} \cdot \lambda(x)](E) = \int_E \frac{1}{\phi(y)} \lambda(x, dy)$.

We set $\psi = \phi^{1/3}$ and define $\mu: T \rightarrow M(S)$ by $\mu(x, E) = \int_E \frac{1}{\psi} \lambda(x, dy) = [\frac{1}{\psi} \cdot \lambda(x)](E)$; since each $\lambda(x)$ vanishes off the non-zeros of ϕ and hence also of ψ the function μ is defined for all $x \in T$ and Borel sets E .

Letting $W = \{t: \phi(t) \geq 1\}$ we have $\psi(t) > \phi(t)$ for $t \in S \setminus W$ and $|\mu|(x, S) = \int_W \frac{1}{\psi(y)} |\lambda|(x, dy) + \int_{S \setminus W} \frac{1}{\psi(y)} |\lambda|(x, dy) \leq |\lambda|(x, W) + \int_{S \setminus W} \frac{1}{\phi(y)} |\lambda|(x, dy) \leq \|\lambda\| + 1$ since $\|\frac{1}{\phi} \cdot \lambda(x)\| \leq 1$ for all $x \in T$.

Hence $\|\mu\| \leq \|\lambda\| + 1 < \infty$ and we will now show that $\mu(\cdot, E)$ is continuous for all Borel sets $E \subset S$.

Let $W_n = \{t \in S: \frac{1}{(n+1)^3} \leq \phi(t) < \frac{1}{n^3}\}$ for $n = 1, 2, \dots$. Then $S = W \cup S \setminus N(\phi) \cup \bigcup_{n=1}^{\infty} W_n$ where $N(\phi) = \{x: \phi(x) > 0\}$ and

$|\lambda|(x, S \setminus N(\phi)) = 0$ as noted above. Furthermore if $t \in W_n$ then $\frac{1}{n+1} \leq \psi(t) < \frac{1}{n}$.

Let $Z_n = \{t \in S : n^3 < \frac{1}{\phi}(t)\}$. Then $W_n \subset Z_n$ and since $\|\frac{1}{\phi} \cdot \lambda(x)\| \leq 1$ it follows that $|\lambda|(x, Z_n) \leq \frac{1}{n^3}$ and consequently $|\lambda|(x, W_n) \leq \frac{1}{n^3}$.

$$\begin{aligned} \text{We now have } \mu(x, E) &= \int_E \frac{1}{\psi(y)} \lambda(x, dy) = \int_{E \cap W} \frac{1}{\psi(y)} \lambda(x, dy) + \\ &+ \int_{S \setminus N(\phi) \cap E} \frac{1}{\psi(y)} \lambda(x, dy) + \int_{E \cap \bigcup_{n=1}^{\infty} W_n} \frac{1}{\psi(y)} \lambda(x, dy) = \\ &= \int_{E \cap W} \frac{1}{\psi(y)} \lambda(x, dy) + \sum_{n=1}^{\infty} \int_{E \cap W_n} \frac{1}{\psi(y)} \lambda(x, dy) = \\ &= \int_S \frac{\chi_{E \cap W}}{\psi}(y) \lambda(x, dy) + \sum_{n=1}^{\infty} \int_S \left(\frac{\chi_{E \cap W_n}}{\psi} \right)(y) \lambda(x, dy). \end{aligned}$$

Now the functions $g_n(t) = \frac{\chi_{E \cap W_n}}{\psi}(t)$ for $\psi(t) \neq 0$ and $g_n(t) = 0$ for $\psi(t) = 0$, belong to $B(S)$ for all n since $\psi(t) \geq \frac{1}{n+1}$ on W_n . Also, $\int_S g_n(y) \lambda(x, dy) = \int_S \frac{\chi_{E \cap W_n}}{\psi}(y) \lambda(x, dy)$. Similar statements hold for $g(t) = \frac{\chi_{E \cap W}}{\psi}(t)$ for $\psi(t) \neq 0$, $g(t) = 0$ for $\psi(t) = 0$.

But since $\lambda(K)$ is weakly compact for all compact subsets K of T then by Theorem 4.5 (2) and (10), we have that $\lambda(g)$, $\lambda(g_n) \in C(T)$ for $n = 1, 2, \dots$. Furthermore, $|\int_{E \cap W_n} \frac{1}{\psi(y)} \lambda(x, dy)| \leq \int_{E \cap W_n} \frac{1}{\psi(y)} |\lambda|(x, dy) \leq \int_{E \cap W_n} (n+1) |\lambda|(x, dy) \leq (n+1) |\lambda|(x, W_n) \leq (n+1) \frac{1}{n^3} = \frac{1}{n^2} + \frac{1}{n^3}$. Consequently the series representation for $\mu(x, E)$ above converges uniformly and each term being the continuous function $\lambda(g_n)$, this means that $\mu(\cdot, E)$ is continuous.

Hence by Theorem 4.3, μ is a kernel satisfying (2) of Theorem 4.5 and so also satisfies (9) of the same theorem.

Hence $\mu(B)$ is weakly relatively compact in $C(T)_\beta$ where B is again the unit ball in $C(S)$. Let $V_\psi = \{f \in C(S) : \|f\psi\| \leq 1\}$. Then V_ψ is a β -neighborhood of 0 since $\psi \in C_0(S)$ and furthermore $\lambda(V_\psi) \subset \mu(B)$. For if $f \in V_\psi$ then $f\psi \in B$ and since $\lambda(f)(x) = \int_S f(y) \lambda(x, dy) = \int_S \frac{f(y)\psi(y)}{\psi(y)} \lambda(x, dy) = \int_S f(y)\psi(y) \frac{1}{\psi(y)} \lambda(x, dy) = \int_S f(y)\psi(y) \mu(x, dy) = \mu(\psi f)(x)$ then $\lambda(V_\psi) \subset \mu(B)$. But since $\mu(B)$ is weakly relatively compact this says $\lambda(V_\psi)$ is weakly relatively compact completing the proof of the theorem.

We are now ready to characterize the β -weakly compact operators on $C(S)_\beta$ into $C(T)_\beta$.

Theorem 5.2: Let A be a linear mapping from $C(S)$ into $C(T)_\beta$. Then A is β -weakly compact if and only if A is continuous from $C(S)_\beta$ into $(C(T), \|\cdot\|)$ and A is weakly compact.

Proof: If A is β -weakly compact then there is a β -neighborhood V of 0 such that $A(V)$ is weakly relatively compact and therefore bounded. Consequently A is continuous from $C(S)_\beta$ into $(C(T), \|\cdot\|)$. Furthermore since V absorbs bounded sets the unit ball $B \subset \alpha V$ for some α so that $A(B)$ is weakly relatively compact.

Conversely, suppose A is continuous from $C(S)_\beta$ into $(C(T), \|\cdot\|)$ and is weakly compact. By Theorem 3.12, A has a kernel representation λ such that $\lambda(T)$ is β -equicontinuous. Since A is weakly compact λ satisfies condition (8) of

Theorem 4.5 and so also (2). But then λ satisfies the conditions of Theorem 5.1 so that there is a β -neighborhood V of 0 such that $A(V) = \lambda(V)$ is weakly relatively compact in $C(T)_\beta$ hence proving that A is β -weakly compact.

We have the following interesting

Corollary 5.3: Let A be a linear mapping from $C(S)$ into $C(T)$ and suppose T is compact. Then A is continuous and weakly compact on $C(S)_\beta$ if and only if A is β -weakly compact.

Proof: For if A is weakly compact and β -continuous its kernel λ satisfies (9) of Theorem 4.5 and so also (6). But since T is compact, (6) says that $\lambda(T)$ is β -equicontinuous and by Theorem 5.2, this makes A β -weakly compact. The converse is clear.

Corollary 5.4: Let A be a linear operator from $C(S)_\beta$ into a Banach space X . Then A is a continuous weakly compact operator into X if and only if A is β -weakly compact.

Proof: Let B be the unit ball in X and let T denote the unit ball B^0 in X^* with the topology $\sigma(X^*, X)$ so that by Theorem 1.3 T is compact. For each $f \in C(S)$ we may consider the image $Af \in X$ as a bounded continuous function on T and, as is well known, the norm of Af as an element of $C(T)$ is $\|Af\|$ as an element of X . For the sake of discussion we denote Af by $A_0 f$ when we consider Af as an element of $C(T)$. Then A_0 is a linear operator from $C(S)_\beta$ into $C(T)$ and is continuous with the norm topology on $C(T)$ if A is continuous into X . Furthermore if $x^* \in B^0 = T$ and λ is the kernel of A_0 then for any

$f \in C(S)$ we have $\langle f, A^*x^* \rangle = \langle Af, x^* \rangle = \langle A_0 f, x^* \rangle = \langle f, A_0^* x^* \rangle = \langle f, \lambda(x^*) \rangle$ so that $\lambda(x^*) = A^*x^*$.

If A is continuous and weakly compact then by Theorem 1.8, $\lambda(T) = A^*(B^0)$ is weakly relatively compact in $M(S)$. This means λ satisfies the conditions of Theorem 4.5 so that A_0 is a weakly compact operator. Since T is compact, then by Corollary 5.3, A_0 is β -weakly compact. This means there is a β -neighborhood V of 0 such that $A_0(V)$ is weakly relatively compact in $C(T)$.

Hence if $x_n = Af_n \in A_0(V)$ then the sequence $\{x_n\}$ has a weak cluster point $x \in C(T)$. In particular, x is a cluster point of $\{x_n\}$ for the topology of pointwise convergence on $T = B^0$. Hence x must be linear on B^0 and hence the restriction of a linear functional on X^* to B^0 . But x , being an element of $C(T)$, is weak* continuous on B^0 and hence belongs to X . But by Theorem 1.4, this means $A(V)$ is $\sigma(X, X^*)$ relatively compact. Since the converse is clear, this completes the proof.

The proof of the above theorem indicates yet another way by which kernel theory may be used to gain information about weakly compact linear maps of $C(S)_\beta$. We will make further use of this idea at the end of this section and now move on to a brief discussion of the kernels of weakly compact linear operators on $C_0(S)$ and $C(S)_\beta$, but before doing so, given an example of a weakly compact operator on $C(S)_\beta$ into a Banach

space X which is not β -weakly compact thus showing the necessity of the continuity hypothesis on A in the statement of Corollary 5.4.

Let T consist of a single point x_0 with the discrete topology and let $S = [0,1]$. Let $M = \{f \in C(S) : f^-(1) = \lim_{x \rightarrow 1^-} f(x) \text{ exists}\}$. Define A on M into $C(T)$ by $[Af](x_0) = f^-(1)$. Then A is no more than a bounded linear functional defined on the subspace M of $C(S)$. By the Hahn-Banach theorem A has an extension to all of $C(S)$; we denote this extension by A . Clearly A is weakly compact but cannot be β -weakly compact since A is not β -continuous. For suppose A were β -continuous. Let $f_n(x) = x^n$ for $x \in [0,1]$, $f(x) = 0$. Then $f_n \rightarrow f$ in the β -topology on $C(S)$ since $\{f_n\}$ is uniformly bounded and compact-open convergent to f . But $f_n \in M$ and so $[Af_n](x_0) = f_n^-(1) = 1$ while $[Af](x_0) = 0$ so that $Af_n \not\rightarrow Af$.

Theorem 5.5: Let A be a linear mapping of $C_0(S)$ into $C(T)_\beta$. Then A is a weakly compact operator on $C_0(S)$ if and only if A is representable by a kernel λ satisfying the equivalent conditions of Theorem 4.5

Proof: If A is weakly compact then A is continuous and so has a kernel representation λ by Theorem 2.20 which satisfies (9) of Theorem 4.5. Conversely if A is representable by a kernel λ satisfying the conditions of Theorem 4.5 then in particular (9) is satisfied so that A is weakly compact.

Theorem 5.6: Let A be a linear mapping of $C(S)_\beta$ into $C(T)_\beta$. Then A is continuous and weakly compact if and only

if A has a kernel representation λ satisfying the conditions of Theorem 4.5.

Proof: If A is continuous then A has a kernel representation λ by Theorem 3.5 and if A is weakly compact then λ satisfies (8) of Theorem 4.5. Conversely suppose A is representable by a kernel λ satisfies the conditions of Theorem 4.5. By (6) of that theorem along with Theorem 3.8 A is continuous and by (8) A is weakly compact.

Theorem 5.7: Let A be a linear mapping of $C(S)_\beta$ into $C(T)_\beta$. Then A is β -weakly compact if and only if A has a kernel representation λ satisfying the equivalent conditions of Theorem 4.5 along with the condition that $\lambda(T)$ be β -equicontinuous.

Proof: If A is β -weakly compact, then, A has a kernel representation λ since A is continuous such that $Af = \lambda(f)$. By Theorem 5.1 $\lambda(T)$ is β -equicontinuous and satisfies (2) of Theorem 4.5. Conversely if λ has those properties then by Theorem 5.1 there is a β -neighborhood V of 0 such that $A(V) = \lambda(V)$ is weakly relatively compact in $C(T)_\beta$.

Theorem 5.8: Let A be a linear mapping of $C_0(S)$ into $C(T)_\beta$. If A is weakly compact then A has a unique extension to a continuous weakly compact operator on $C(S)_\beta$. Furthermore if the range of A is a subset of $C_0(T)$ this extension is β -weakly compact.

Proof: Since A is weakly compact then A is representable by a kernel λ satisfying all the conditions of Theorem 4.5

according to Theorem 5.5. Extend A to $C(S)$ by the formula $Af = \lambda(f) = A^{**}f$ for $f \in C(S)$. By Theorem 5.6 this extension defines a continuous weakly compact operator on $C(S)_\beta$. By Theorem 3.11 the extension is unique. Finally, if the range of A is contained in $C_0(T)$, then by Theorem 1.8, A^* maps equicontinuous sets of $M(T)$ (as subsets of the dual of $C_0(T)$) into weakly relatively compact subsets of $M(S)$. But $\{\overset{\circ}{x}: x \in T\}$ is $C_0(T)$ -equicontinuous and $A^*\{\overset{\circ}{x}: x \in T\} = \lambda(T)$ so that $\lambda(T)$ is weakly relatively compact and certainly β -equicontinuous. Consequently the extension of A is β -weakly compact.

It should be clear from the proof above that a weakening of the hypothesis for the β -weak compactness of the extension of A may be possible since it was not used in its full strength in the proof. In this respect see Theorem 3.12.

We have one final result to extract from our knowledge of kernels.

Theorem 5.9: Let A be a linear operator from $C_0(S)$ or $C(S)_\beta$ into $C(T)_\beta$ which maps real functions on S into real functions on T . If A is continuous and weakly compact and if the kernel of A satisfies any one of the conditions of Theorem 4.8, then $A = A^+ - A^-$, where A^+ and A^- are continuous weakly compact operators which map positive functions into positive functions. If λ is the kernel of A and we set $(|A|f)(x) = \int_S f(y) |\lambda|(x, dy)$ then $|A|$ is continuous and weakly compact and $A^+ = (A + |A|)/2$, $A^- = (|A| - A)/2$.

Proof: The proof is clear from a consideration of Theorem 4.8 and Theorems 5.5 and 5.6.

Before studying the compact operators on $C(S)_B$ we prove a final result on weakly continuous kernels. We include it only for the sake of completeness for its proof is not difficult given Theorem 1.19 and is very much like the proof given in Edwards [13,p.665] characterizing the weakly compact operators on $C(S)$ with the compact-open topology.

Theorem 5.10: Let $\lambda: T \rightarrow M(S)$ be a kernel satisfying the equivalent conditions of Theorem 4.5 and suppose that T is σ -compact. Then there is a measure $\mu \in M(S)^+$ and a complex-valued function k on $S \times T$ such that $k(\cdot, x) \in L^1(\mu)$ for all $x \in T$ and $\lambda(x, E) = \int_E k(y, x) \mu(dy)$ for all $x \in T$ and Borel sets E .

Proof: Since T is σ -compact and locally compact we can write $T = \bigcup_{n=1}^{\infty} K_n$ where each set K_n is compact and contained in the interior, K_{n+1}^0 , of K_n .

Since K_n is compact, then by (2) of Theorem 4.5, the set $\lambda(K_n)$ is weakly relatively compact in $M(S)$ and so by Theorem 1.19 there is a measure $\mu_n \in M(S)^+$, and for each $x \in K_n$ a function $f_n(\cdot, x) \in L^1(\mu)$, such that $\lambda(x, E) = \int_E f_n(y, x) \mu_n(dy)$, for all Borel sets E and all $x \in K_n$. Let $g_n(y, x) = \|\mu_n\| f_n(y, x)$ and $\nu_n = \mu_n / \|\mu_n\|$ if $\|\mu_n\| \neq 0$, $\nu_n = 0$ otherwise. Then $\lambda(x, E) = \int_E g_n(y, x) \nu_n(dy)$ and $\|\nu_n\| = 1$.

Let $\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_n(E)$. Then $\mu \in M(S)^+$ and each measure ν_n is absolutely continuous with respect to μ so that by the

Radon-Nikodym theorem there is a function $h_n \in L^1(\mu)$ such that $\nu_n(E) = \int_E h_n(y) \mu(dy)$ for all Borel sets E and consequently $\lambda(x, E) = \int_E g_n(y, x) h_n(y) \mu(dy)$ for all Borel sets E and $x \in K_n$.

Let $k_n(y, x) = g_n(y, x) h_n(y)$ and let $x \in K_n \cap K_m$. Then $\int_E k_n(y, x) \mu(dy) = \lambda(x, E) = \int_E k_m(y, x) \mu(dy)$ for all Borel E so that $k_n(\cdot, x) = k_m(\cdot, x)$ almost everywhere μ for any n and m . Since the collection $\{k_n\}$ is countable this implies that the function $k(y, x) = k_n(y, x)$ of $x \in K_n$ is well-defined almost everywhere μ for each $x \in T$, and $k(\cdot, x) \in L^1(\mu)$ for all $x \in T$. Finally, it is clear that $\lambda(x, E) = \int_E k(x, y) \mu(dy)$, completing the proof.

The corollaries and applications to operator representations which can be drawn from Theorem 5.10 should be clear. One only has to consider its application in conjunction with Theorems 5.6 through 5.8.

We now turn to a study of the compact and β -compact operators on $C(S)$ with a view towards gaining results analogous to those for weakly compact operators. We obtain first the analogue of Theorem 5.1.

Theorem 5.11: Let $\lambda: T \rightarrow M(S)$ be a kernel. Then there is a β -neighborhood V of zero such that $\lambda(V)$ is relatively compact in $C(T)_\beta$ if and only if $\lambda(T)$ is β -equicontinuous and $\lambda(K)$ is compact in $M(S)$ for each compact subset K of T .

Proof: If $\lambda(V)$ is β -relatively compact then by Theorem 5.1 the set $\lambda(T)$ is β -equicontinuous. Furthermore it is clear that λ satisfies (3) of Theorem 4.13 and so also (2).

Conversely, suppose $\lambda(T)$ is β -equicontinuous and $\lambda(K)$ is compact for each compact subset K of T . Since $\lambda(T)$ is β -equicontinuous, we can proceed exactly as in the proof of Theorem 5.1 to obtain a function $\phi \geq 0$ in $C_0(S)$ such that $\|\frac{1}{\phi} \cdot \lambda(x)\| \leq 1$ for all $x \in T$. We define $\psi = \phi^{1/3}$ as before and again set $\mu(x, E) = \int_S \frac{1}{\psi(y)} \lambda(x, dy)$. We define, as in Theorem 5.1, the sets W and W_n . It follows that μ is a kernel by the proof given there.

We wish to show that μ satisfies (1) of Theorem 4.13.

As before $\mu(x, E) = \int_{E \cap W} \frac{1}{\psi(y)} \lambda(x, dy) + \sum_{n=1}^{\infty} \int_{E \cap W_n} \frac{1}{\psi(y)} \lambda(x, dy)$ for all Borel sets E . If B denotes any one of the sets W or W_n and we set $\nu_B(x, E) = \int_{E \cap B} \frac{1}{\psi(y)} \lambda(x, dy)$ then since λ satisfies (2) of Theorem 4.13 and $\|\nu_B(x) - \nu_B(x_0)\| \leq a_B \|\lambda(x) - \lambda(x_0)\|$, where $a_B = 1$ if $B = W$, $a_B = n+1$ if $B = W_n$, then by (1) of Theorem 4.13 the function $x \rightarrow \nu_B(x)$ is a continuous mapping of T into $M(S)$.

Since $\mu(x) = \nu_W(x) + \sum_{n=1}^{\infty} \nu_{W_n}(x)$ for each $x \in T$ we will have that $x \rightarrow \mu(x)$ is norm continuous provided the convergence of the series on the right is uniform in the norm topology of $M(S)$.

We have $\|\nu_{W_n}(x)\| = \sup\{|\int_S f(y) \nu_{W_n}(x, dy)| : \|f\| \leq 1\} = \sup\{|\int_{W_n} f(y)/\lambda(y) \lambda(x, dy)| : \|f\| \leq 1\} \leq \sup\{|\int_{W_n} \frac{f(y)}{\lambda(y)} \lambda(x, dy)| : \|f\| \leq 1\} \leq (n+1) |\lambda|(x, W_n) \leq (n+1) 1/n^3$ by the same arguments for the proof of Theorem 5.1.

Consequently for all $x \in T$ $\left\| \sum_{k=n}^m v_{W_k}(x) \right\| \leq \sum_{k=n}^m \|v_{W_k}(x)\|$
 $\leq \sum_{k=n}^m \frac{n+1}{n}$ and the convergence is uniform and μ satisfies (1)
of Theorem 4.13.

We complete the proof by exactly the same arguments as in Theorem 5.1 making use of the fact that μ now satisfies (3) of Theorem 4.13.

The result above yields the analogue of Theorem 5.2, stated as follows:

Theorem 5.12: Let A be a linear mapping of $C(S)_\beta$ into $C(T)_\beta$. Then A is β -compact if and only if A is compact and continuous with the norm topology on $C(T)$.

It is clear that Corollary 5.3 and theorems 5.5, 5.6, 5.7, 5.8 and 5.9 remain valid with the words "weakly compact" replaced by "compact" and with "Theorem 4.5" replaced by "Theorem 4.13". It is also clear that Theorem 5.10 remains valid and can be strengthened slightly. For since $x \mapsto \lambda(x)$ is norm continuous then given $\epsilon > 0$ and $x_0 \in T$ there is a neighborhood U of x_0 such that $\int_S |k(y, x) - k(y, x_0)| \mu(dy) = \|\lambda(x) - \lambda(x_0)\| < \epsilon$ for all $x \in U$.

The relationship between the kernel λ and the function k is yet to be determined, to the best of our knowledge. Certain conditions on k trivially imply that λ is either weakly or norm continuous but according to [9, p.490] necessary conditions on k are not known.

We prove the analogue of Corollary 5.4.

Theorem 5.12: Let A be a linear mapping of $C(S)_\beta$ into a Banach space X . Then A is continuous and compact if and only if A is β -compact.

Proof: We proceed as in the proof of Corollary 5.4 taking T to be the polar of the unit ball in X with the weak* topology and letting $A_\circ f$ represent Af considered as a continuous function on T and $\lambda(x^*) = A^*x^*$ for $x^* \in T = B^\circ$. If A is a compact operator then $A(B)$ is relatively compact in X where B is the unit ball in $C(S)$. But the norm topology on X is the supremum norm topology on X considered as a subset of $C(T)$, and consequently $A_\circ(B)$ is relatively compact in $C(T)$ so that λ , being the kernel of A_\circ , satisfies the equivalent conditions of Theorem 4.13, and additionally $\lambda(T) = A^*(B^\circ)$ is β -equicontinuous by Lemma 3.4, so that by Theorem 5.12, A_\circ is β -compact and this says that A is β -compact. The converse being clear, this completes the proof.

Actually one may more formally describe the idea by which kernel theory may be applied to operators with range in a normed vector space but we do not do so since the proof of the above theorem makes the method of application abundantly clear. If the range of the operator A is a locally convex space X , one may take T to be the polar of a neighborhood V in X^* with the topology $\sigma(X^*, X)$ and gain the representation $[Af](x^*) = \lambda(f)(x^*)$ for $x^* \in V^\circ$. There are, however, some obvious limits to just how applicable this method can actually be.

This concludes our study of compact and weakly compact operators on $C(S)_\beta$. Certain interesting questions remain open however. A consideration of Theorem 5.9 causes one to ask if operators A for which the conclusions of that theorem holds can be intrinsically characterized? The example preceding Corollary 4.7 along with a consideration of Theorem 5.10 may be helpful in this respect. One might also ask if the product of weakly compact operators on $C(S)_\beta$ is compact as is the case when S is compact.

CHAPTER VI

SEMIGROUPS OF OPERATORS ON $C_0(S)$ AND $C(S)_\beta$

Let $\{\lambda_t: t \geq 0\}$ be a transition function on S . That is, for each $t \geq 0$, $\lambda_t: S \rightarrow M(S)$, $\lambda_t(\cdot, E)$ is Borel measurable for each Borel set E and $\lambda_{t+u}(x, E) = \int_S \lambda_t(y, E) \lambda_u(x, dy)$.

For $f \in B(S)$ set $[T_t f](x) = \lambda_t(f)(x) = \int_S f(y) \lambda_t(x, dy)$. Then $T_t f \in B(S)$ by the proof of Theorem 2.11, and $[T_t(T_u f)](x) = \lambda_t(T_u f)(x) = \int_S [T_u f](y) \lambda_t(x, dy) = \int_S \int_S f(z) \lambda_u(y, dz) \lambda_t(x, dy) = \int_S f(y) \lambda_{t+u}(x, dy)$ by Theorem 2.13 with $\lambda = \lambda_u$, $\mu = \lambda_t$ and $\nu = \lambda_{t+u}$. Hence $\{T_t: t \geq 0\}$ is a semigroup of operators on $B(S)$ and $\|T_t\| = \|\lambda_t\|$.

By Theorem 3.8, $T_t(C(S)) \subset C(S)$ provided λ_t is a kernel satisfying condition E, in which case, T_t is also a continuous operator from $C(S)_\beta$ into $C(S)_\beta$. In this case $\{T_t: t \geq 0\}$ is a semigroup of operators on $C(S)_\beta$; clearly $\{T_t: t \geq 0\}$ is a semigroup on $C_0(S)$ if and only if $\lambda_t(f) \in C_0(S)$ for $f \in C_0(S)$. Finally the operators T_t are always norm continuous.

Theorem 6.1: Let $\{\lambda_t: t \geq 0\}$ be a transition function on $M(S)$. The formula $T_t f = \lambda_t(f)$ defines a semigroup of continuous linear operators in $C(S)_\beta$ provided that there is a number $a > 0$ such that λ_t is a kernel on S for all $t < a$ and λ_t satisfies condition E for all $t < a$.

Proof: Our earlier remarks show that $\{T_t: t \geq 0\}$ is a semigroup on $B(S)$ and furthermore under the given hypotheses we see that $T_t(C(S)) \subset C(S)$ for all $t < a$. If $u \in [a, 2a)$ then

$u = t + h$ for $t, h < a$ and for $f \in C(S)$ one has $T_u f = T_t T_h f \in C(S)$ so that by induction $T_t C(S) \subset C(S)$ for all $t \geq 0$. Furthermore, the hypotheses imply by Theorem 3.8 that the operators T_t are β -continuous for $t < a$ so that T_u is also β -continuous for $u \in [a, 2a)$, and again by induction, T_t is continuous for all $t \geq 0$.

From the viewpoint of the study of Markov processes the following corollary is of interest.

Corollary 6.2: Suppose $\{\lambda_t : t \geq 0\}$ is a transition function on a paracompact space S , and suppose that the formula $T_t f = \lambda_t(f)$ defines a linear transformation T_t on $C(S)$ into itself for all t less than some number $a > 0$. Then $\{T_t : t \geq 0\}$ is a semigroup of β -continuous operators on $C(S)$.

Proof: Since $T_t f \in C(S)$ for all $f \in C(S)$ with $t < a$ then by Corollary 3.10 T_t is a β -continuous operator on $C(S)$ for all $t < a$ and additionally, by Theorem 3.5, λ_t satisfies condition E for all $t < a$ and the conclusion follows from Theorem 6.1.

The analogous results for $C(S)_\beta$, are clear and we do not state them.

We now give the converse of Theorem 6.1 which is known for first countable spaces and positive operators. See Dynkin [11, p.52].

Theorem 6.3: Let $\{T_t : t \geq 0\}$ be a semigroup of operators in $C_0(S)$, $C(S)_\beta$ or $C(S)_{\beta, \cdot}$. The semigroup $\{T_t : t \geq 0\}$ is uniquely defined by some transition function $\{\lambda_t : t \geq 0\}$ such that for each $t \geq 0$, λ_t satisfies condition E or E' according

to the space $C(S)_\beta$ or $C(S)'_\beta$, on which each T_t is continuous.

Proof: Theorems 2.21 and 3.6 yield kernels λ_t , for each $t \geq 0$, such that $T_t^{**}f = \lambda_t(f)$ for all $f \in B(S)$. Since $T_{t+u}^{**} = T_t^{**}T_u^{**}$, letting $f = \chi_E$ yields the Chapman-Kolmogorov equation $\lambda_{t+u}(x, E) = [T_{t+u}^{**} \chi_E](x) = (T_t^{**}[T_u^{**} \chi_E])(x) = \int_S \lambda_u(y, E) \lambda_t(x, dy)$ so that $\{\lambda_t; t \geq 0\}$ is a transition function. Applications of Theorem 3.5 yield the remaining conclusions.

If each operator T_t is a bounded operator with norm 1 and also positive then each measure $\lambda_t(x)$ is a probability measure which is regular on S . It then follows from the work of Kolmogorov [21], along with a generalization of it found in Neveu [27], that the transition function $\{\lambda_t; t \geq 0\}$ is the transition function of a Markov process. That is, there is a space Ω with σ -algebras M_t , trajectories x_t on Ω into S , and probability measures P_x for each $x \in S$, such that $\lambda_t(x, E) = P_x(x_t^{-1}(E))$ for all Borel sets E , so that $[T_t f](x) = \int_\Omega f(x_t(\omega)) P_x(d\omega)$ for all $f \in C(S)$ and $x \in S$. Consequently $[T_t f](x)$ is the expectation of $f \circ x_t$ with respect to the probability P_x . It follows that the considerable theory of Markov processes is now applicable to the study of such semigroups on $C(S)_\beta$.

Conversely, given a Markov process with transition function λ_t , or any transition function λ_t , we have seen that the induced semigroup T_t consists of β -continuous operators on $C(S)$ provided that S is paracompact and T_t leaves $C(S)$ invariant. In order to apply the general theory of semigroups on locally convex spaces found in Yosida [33] one requires that

the operators $\{T_t: t \geq 0\}$ be equicontinuous. In general this may not be the case. For example consider the translation semigroup $[T_t f](x) = f(x+t)$ on $S = [0, \infty)$ and $f \in C(S)$. Let $g(x) = e^{-x}$ so that $g \in C_0(S)$ and suppose there is a $\psi \in C_0(S)$ such that $\|f\psi\| \leq 1$ implies $\|gT_t f\| \leq 1$. Let $[0, a]$ be an interval such that if $|\psi(x)| > 1/e$ then $x \in [0, a]$. Let f be a function identically 0 on $[0, a]$ and reaching a maximum of e at some point $a + t_0$. Then $\|f\psi\| \leq 1$ and $|g(0)[T_{t_0} f](0)| = |f(0 + t)|$ is assumed to be less than or equal to one for all $t \geq 0$. Choosing $t = a + t_0$ yields a contradiction.

Our next task is to characterize those semigroups which are β -equicontinuous on bounded intervals and are strongly continuous on $C_0(S)$; i.e., such that the subspace X_0 defined in Chapter I contains $C_0(S)$.

We begin with a generalization of a result found in [11, p.36].

Lemma 6.4: Let X be a locally convex space and let $\{T_t: t \geq 0\}$ be a semigroup of linear operators in X which is equicontinuous on an interval $[0, a)$, for some $a > 0$. Let $X_0 = \{x \in X: T_t x \rightarrow x \text{ as } t \rightarrow 0\}$, and let $X_1 = \{x \in X: T_t x \rightarrow x \text{ } \sigma(X, X^*) \text{ as } t \rightarrow 0\}$. Then $X_0 = X$ if and only if $X_1 = X$.

Proof: Clearly $X_0 \subset X_1$ and we need only show that $X_1 = X$ implies $X_0 = X$.

Fix $x^* \in X^*$ and let $f(t) = \langle T_t x, x^* \rangle$ for a fixed $x \in X_1 = X$. Then $f(t+h) - f(t) = \langle T_{t+h} x - T_t x, x^* \rangle = \langle T_h x - x, T_t^* x^* \rangle$ and since $x \in X_1$ then $f(t+h) \rightarrow f(t)$ as $h \rightarrow 0$, $h \geq 0$. Hence f is

continuous from the right and consequently is Lebesgue measurable. That is the function $t \rightarrow T_t x$ is weakly measurable.

We now claim that the function $t \rightarrow T_t x$ is almost separably valued; i.e., its values all lie in a separable subspace of X save for all t belonging to a set of Lebesgue measure zero. Let L denote the closure of the subspace consisting of all vectors of the form $\sum_{i=1}^n a_i T_{r_i} x$ where r_i is rational and a_i is a complex number with rational real and imaginary parts. Then L is a closed subspace of X , and moreover L is separable. If for some t , $T_t x \notin L$ then by the Hahn-Banach theorem there is a vector $x^* \in X^*$ such that $\langle y, x^* \rangle = 0$ for all $y \in L$ and $\langle T_t x, x^* \rangle = 1$. If $\{r_n\}$ is a decreasing sequence of rational numbers converging to t then $T_{r_n} x \in L$ and $\langle T_t x - T_{r_n} x, x^* \rangle \rightarrow 0$. But $\langle T_{r_n} x, x^* \rangle = 0$ while $\langle T_t x, x^* \rangle = 1$. Consequently $t \rightarrow T_t x$ is separately valued.

It now follows from [29, Remark 1] and the hypothesis of equicontinuity that $t \rightarrow T_t x$ is continuous at all points $t > 0$. From this and the semigroup property one has that $X_0 \supset \{T_t x : t > 0, x \in X\}$.

Suppose $x \in X_1$ but $x \notin X_0$. Since by (1) of Theorem 1.23 the set X_0 is a closed subspace of X there exists an $x^* \in X^*$ such that $\langle y, x^* \rangle = 0$ for all $y \in X_0$ and $\langle x, x^* \rangle = 1$. But since $x \in X_1$ then $\langle T_t x, x^* \rangle \rightarrow \langle x, x^* \rangle$ and this is a contradiction since $\langle T_t x, x^* \rangle = 0$ for all $t > 0$. Consequently $X_1 \subset X_0 \subset X_1$ and the proof is complete.

Our principle result is the following:

Theorem 6.5: Let $\{T_t : t \geq 0\}$ be a semigroup of β -continuous operators with transition function $\{\lambda_t : t \geq 0\}$. Let $a \in (0, \infty)$ and $T^a = [0, a] \times S$. Suppose $N_a = \sup\{\|\lambda_t\| : t \leq a\} < \infty$. Consider the following statements.

- (1) $\{T_t : t \leq a\}$ is β -equicontinuous and for all $f \in C_0(S)$, $T_t f \rightarrow f$ pointwise on S as $t \rightarrow 0$.
- (2) The mapping $\mu : T^a \rightarrow M(S)$ defined by $\mu((t, x), E) = \lambda_t(x, E)$ satisfies condition E and $T_t f \rightarrow f$ as $t \rightarrow 0$ in the β topology on $C(S)$ for all $f \in C(S)$.
- (3) Same as (2) with the β -topology replaced by the compact-open topology on $C(S)$.
- (4) The mapping μ satisfies E and the mapping $(t, x) \rightarrow [T_t f](x)$ defines a continuous function on T^a for all $f \in C(S)$.
- (5) The same as (4) with $C(S)$ replaced by $C_0(S)$.
- (6) The formula $[Bf](t, x) = [T_t f](x)$ defines a continuous linear operator on $C(S)_\beta$ into $C(T^a)_\beta$.
- (4') The mapping $(t, x) \rightarrow [T_t f](x)$ is a continuous function on T^a for all $f \in C(S)$.
- (6') $B(C(S)) \subset C(T^a)$.

The conclusions are: (1) through (6) are equivalent, and if S is paracompact, then (4) and (6) may be replaced by (4') and (6').

Proof: Suppose (1) holds. We show that (2) follows. If Q is a compact subset of T^a , then $Q \subset [0, a] \times K$ for some

compact subset K of S . Let $\psi \in C_0(S)$, such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on K . Since $\{T_t : t \leq a\}$ is β -equicontinuous there is a neighborhood V of 0 in $C(S)$ such that if $f \in V$ then $\|\psi T_t f\| \leq 1$. If $(t, x) \in Q$ then $x \in K$ so that $|[T_t f](x)| = |\psi(x)[T_t f](x)| \leq 1$ and this says $1 \geq |\lambda_t(f)(x)| = |\mu(f)(t, x)|$ for all $(t, x) \in Q$. Hence $\mu(Q) \subset V^0$, so that by definition μ satisfies E.

Now fix $x \in S$ and let $v(t, E) = \lambda_t(x, E)$ for $t > 0$ and $v(0, E) = \overset{0}{x}(E)$. From what we have just shown, $\{v_t : t \leq a\}$ satisfies E and from (1) $v(f)(t) = [T_t f](x) + f(x) = v(f)(0)$ as $t \rightarrow 0$ for all $f \in C_0(S)$. By Corollary 3.3, $v(f)(t) \rightarrow v(f)(0)$ for all $f \in C(S)$. Consequently $T_t f \rightarrow f$ pointwise for all $f \in C(S)$. Since $N_a < \infty$ the set $\{T_t f : t \leq a\}$ is uniformly bounded and consequently $T_t f \rightarrow f$ weakly as $t \rightarrow 0$ for all $f \in C(S)$. That is, the set X_1 of Lemma 6.4 is $C(S)$. Consequently $X_0 = C(S)$ which means that $T_t f \rightarrow f$ in the β -topology on $C(S)$ for all $f \in C(S)$, proving (2).

Clearly (2) implies (3). Given (3), let us prove (4). Since $\|T_t f\| \leq N_a \|f\|$ for $t \leq a$ then $\{T_t f : t \leq a\}$ is uniformly bounded and since the β and compact open topologies agree on bounded sets then $T_t f \rightarrow f$ in $C(S)_\beta$ as $t \rightarrow 0$. Hence if $t_n \rightarrow t_0 \in [0, a)$ then $T_{t_n - t_0} f \rightarrow f$ and since T_{t_0} is β -continuous then $T_{t_n} f = T_{t_0} T_{t_n - t_0} f \rightarrow T_{t_0} f$ in $C(S)_\beta$. Furthermore if $x \in S$ then choosing a neighborhood U of x with compact closure there exists an integer N such that $|[T_{t_n} f](y) - [T_{t_0} f](y)| < \epsilon$ for $n \geq N$ and all $y \in U$. It follows that if $x_\alpha \rightarrow x$ then $[T_{t_n} f](x_\alpha) \rightarrow [T_{t_0} f](x)$.

Now let $t_n \rightarrow t_0$ and $x_\alpha \rightarrow x$ with U chosen as above. Choose α_0 such that if $\alpha \geq \alpha_0$ then $x_\alpha \in U$. Since μ satisfies E on T^a and \bar{U} is compact there is a β -neighborhood V of 0 such that $|\mu(g)(t, y)| < \epsilon/2$ for all $(t, y) \in [0, a] \times \bar{U}$. Since $t_0 - t_n \rightarrow 0$ then given $f \in C(S)$ there is an N such that $T_{t_0 - t_n} f - f \in V$ for all $n \geq N$. Consequently $|\mu(T_{t_0 - t_n} f - f)(t, y)| < \epsilon/2$ for all $t \leq a, y \in U$. Choose $\beta_0 \geq \alpha_0$ such that $|[T_{t_0} f](x_\alpha) - [T_{t_0} f](x)| < \epsilon/2$ for $\alpha \geq \beta_0$.

$$\begin{aligned} & \text{If } n \geq N \text{ and } \alpha \geq \beta_0 \text{ then } |[T_{t_n} f](x_\alpha) - [T_{t_0} f](x)| \\ & \leq |[T_{t_n} f](x_\alpha) - T_{t_n} [T_{t_0 - t_n} f](x_\alpha)| + |[T_{t_n} f](x_\alpha) - [T_{t_n} f](x)| \\ & \leq |\mu(f - T_{t_0 - t_n} f)(t_n, x_\alpha)| + |[T_{t_n} f](x_\alpha) - [T_{t_n} f](x)| \\ & < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence for any sequence $t_n \rightarrow t_0$ and net $x_\alpha \rightarrow x$ we have $[T_{t_n} f](x_\alpha) \rightarrow [T_{t_0} f](x)$ proving (4).

Clearly (4) implies (5). If (5) holds then to say that $(t, x) \rightarrow [T_t f](x)$ is continuous on T^a for $f \in C_0(S)$ merely means that μ is a kernel on T^a , since $\mu(f)(t, x) = [T_t f](x)$, and since μ satisfies E and $Bf = \mu(f)$, then by Theorem 3.8, (6) holds.

Clearly (6) implies that $T_t f \rightarrow f$ pointwise for $f \in C_0(S)$. Furthermore, since B is continuous on $C(S)_\beta$ into $C(T^a)_\beta$ then $\{T_t : t \leq a\}$ is β -equicontinuous. For let V_ψ be a β -neighborhood of 0 in $C(S)$ and let $V = \{f \in C(T^a) : f_t = f(t, \cdot) \in V_\psi \text{ for } t \leq a\}$. Then $V = V_\phi$ is a β -neighborhood of 0 in $C(T^a)$ where $\phi(t, x) = \psi(x)$ and hence there is a β -neighborhood U of 0 in $C(S)$ such

that $B(U) \subset V$. That is, $f \in U$ implies $[Bf](t, \cdot) = T_t f \in V$ so that $T_t(U) \subset V$ for all $t \leq a$ and $\{T_t : t \leq a\}$ is β -equicontinuous.

Finally, if S is paracompact both (4') and (6') imply $Bf = \mu(f)$ takes $C(S)$ into $C(T)$. By Theorem 3.2 μ satisfies condition E, so that (6) holds by Theorem 3.8 and hence (4') and (6') are equivalent to (4) and (6) respectively.

As a consequence of this theorem we have the following important corollary.

Theorem 6.6: Let $\{T_t : t \geq 0\}$ be a semigroup of continuous operators in $C(S)_\beta$ with transition function $\{\lambda_t : t \geq 0\}$. Suppose that $M = \sup\{\|\lambda_t\| : t \geq 0\} < \infty$ and that $\{T_t : t \geq 0\}$ satisfies any one of the conditions of Theorem 6.5 for some $a > 0$. Then for every $\alpha > 0$ the semigroup $S_t^\alpha = e^{-\alpha t} T_t$ is a strongly continuous semigroup which is β -equicontinuous on $[0, \infty)$.

Proof: Since by (2) of Theorem 6.5, $\{T_t : t \geq 0\}$ is strongly continuous on $C(S)_\beta$ (i.e., $X_0 = C(S)_\beta$), so is S_t^α for each $\alpha > 0$. Let $T = [0, \infty) \times S$ and let $\mu : T \rightarrow M(S)$ be defined by $\mu((t, x), E) = \lambda_t(x, E)$. Since $\{T_t : t \geq 0\}$ satisfies the condition (1) of Theorem 6.5 for some $a > 0$ this condition is also satisfied for all $a > 0$ by the semigroup property. Consequently the remaining conditions of Theorem 6.5 are valid for all $a > 0$. Since every compact subset K of T is contained in a set $[0, a] \times Q$ for some $a > 0$ and some compact set $Q \subset S$ it follows from (2) that $\mu(K)$ is β -equicontinuous so that μ

satisfies condition E on T. Furthermore from (4) μ is a kernel on T since $M \geq \|\mu\|$. Hence the formula $Bf = \mu(f)$ defines a continuous linear operator on $C(S)_\beta$ into $C(T)_\beta$ by Theorem 3.8.

Let $\psi \in C_0(S)$ and define $\phi \in C_0(T)$ by $\phi(t, x) = e^{-\alpha t} \psi(x)$ for a fixed $\alpha > 0$. Then there is a β -neighborhood V of 0 in $C(S)$ such that $B(V) \subset V_\phi$. That is, if $f \in V$ then $\|\phi Bf\| \leq 1$ or $\sup\{|\psi(x)e^{-\alpha t}[T_t f](x)| : (t, x) \in [0, \infty) \times S\} \leq 1$ for all $f \in V$. But this says $\|\psi S_t^\alpha f\| \leq 1$ for every $t \geq 0$ and $f \in V$ completing the proof.

The significance of Theorem 6.6 resides in the fact that a semigroup $\{T_t\}$ satisfying the condition (1) of Theorem 6.5 (which is entirely analogous to the usual assumption for semigroups on Banach spaces) can be transformed, by multiplication by $e^{-\alpha t}$, into a semigroup satisfying the hypotheses of the general theory found in Yosida [33], provided that $\sup\|T_t\| < \infty$. This latter condition is always satisfied when the semigroup arises from a Markov process; that is, when the transition function consists of probability measures.

Corollary 6.7: Let $\{T_t : t \geq 0\}$ be as in Theorem 6.7 and $M < \infty$. If S is paracompact and the mapping $(t, x) \rightarrow [T_t f](x)$ is continuous on $[0, a) \times S$ for all $f \in C(S)$ and some fixed $a > 0$ then $\{e^{-\alpha t} T_t : t \geq 0\}$ is a strongly continuous semigroup on $C(S)_\beta$ which is equicontinuous on $[0, \infty)$ for all $\alpha > 0$.

If each measure $\lambda_t(x)$ is a probability measure it is shown in Dynkin [11, p.54] that $T_t f \rightarrow f$ pointwise on S for all $f \in C(S)$ if and only if λ_t is stochastically continuous; that

is, $\lambda_t(x, U) \rightarrow 1$ as $t \rightarrow 0$ for all $x \in U$ and all open subsets U of S .

Theorem 6.8: Let $\{\lambda_t: t \geq 0\}$ be a transition function consisting of probability measures. If $\lambda_t(x, E) \rightarrow 1$ as $t \rightarrow 0$ for $x \in E$, where E is an open σ -compact subset of S or a closed strict G_δ , then the adjoint semigroup $(T_t^*\mu)(E) = \int_S \lambda_t(x, E) \mu(dx)$ is strongly continuous on $M(S)$ and conversely.

Proof: If C is a closed strict G_δ set and $x \in C$ then $\lambda_t(x, S \setminus C) \rightarrow 1$ and since $\lambda_t(x, S) = 1$ this means $\lambda_t(x) \xrightarrow{0} x$ on closed strict G_δ sets as $t \rightarrow 0$. Since $|\lambda_t(x, C)| \leq 1$ for all $x \in S$, if $\mu \in M(S)$ then $\int_S \lambda_t(x, C) \mu(dx) \rightarrow \int_S x(C) \mu(dx) = \mu(C)$ for all closed strict G_δ sets C . Hence $T_t^*\mu \rightarrow \mu$ on closed strict G_δ sets and by (2) of Corollary 4.11 this means $T_t^*\mu \rightarrow \mu$ weakly. By Lemma 6.4, $\{T_t^*\}$, being equicontinuous on $M(S)$, is strongly continuous. The converse is clear.

We note that this theorem remains valid if the condition that the $\lambda_t(x)$ be probability measures is replaced by the assumption that $\sup\{\|\lambda_t\|: t \leq a\} < \infty$ for some $a > 0$.

For our final result we turn to a consideration of an important class of semigroups, those induced by a semigroup of maps on the space S . We suppose that for each $t \geq 0$ a continuous mapping $\psi_t: S \rightarrow S$ exists such that $\psi_{t+u} = \psi_t \psi_u$ for all $u, t \geq 0$, with $\psi_0(x) = x$ for all $x \in S$. The formula $T_t f = f \circ \psi_t$ defines a semigroup on $C(S)_\beta$ with transition function $\lambda_t(x) = \psi_t^0(x)$. Clearly the semigroup consists of β -continuous operators with $\|\lambda_t(x)\| = 1$.

Dorroh [8] has shown that multiplication by $e^{-\alpha c}$ makes such a semigroup β' -equicontinuous on $[0, \infty)$. We improve this to β -equicontinuity and give a new condition (condition (4) below) on the maps $\{\psi_t\}$ which is equivalent to this; the remaining conditions are due to Dorroh.

Theorem 6.9: With the maps $\{\psi_t: t \geq 0\}$ and semigroup $\{T_t: t \geq 0\}$ defined as above, the following are equivalent:

- (1) For each $\alpha > 0$, the semigroup $\{e^{-\alpha t} T_t\}$ is β -equicontinuous on $[0, \infty)$ and strongly continuous on $C(S)_\beta$.
- (2) The map $t \rightarrow \psi_t(x)$ is continuous on $[0, \infty)$ for all $x \in S$.
- (3) The map $(t, x) \rightarrow \psi_t(x)$ is continuous on $[0, a) \times S$ for all $a > 0$.
- (4) $\psi_t(x) \rightarrow x$ as $t \rightarrow 0$ and for all $a > 0$, the set $\bigcup_{t \leq a} \psi_t(Q)$ is compact in S for each compact subset Q of S .
- (5) $\psi_t(x) \rightarrow x$ as $t \rightarrow 0$ and $\{T_t: t \leq a\}$ is β -equicontinuous for some $a > 0$.

Proof: To see that (1) implies (2), fix $x \in S$ and $t \in [0, \infty)$. If U is a neighborhood of $\psi_t(x)$ with compact closure there is a function $f \in C_0(S)$ such that $f(\psi_t(x)) = 1$ and $f \equiv 0$ on $S \setminus U$. By (2) of Theorem 1.23, $t \rightarrow T_t f$ is continuous on $[0, \infty)$, so that there is a $\delta > 0$ such that if $|u - t| < \delta$ then $\|\psi(T_t f - T_u f)\| < 1/2$ where we take $\psi = f$. In particular then, $1/2 > |\psi(x)f(\psi_t(x)) - \psi(x)f(\psi_u(x))| = |1 - f(\psi_u(x))|$ so that $f(\psi_u(x)) > 0$ and

consequently $\psi_u(x) \in U$. This proves (2).

That (2) implies (3) follows immediately from [8, Theorem 2.2].

To see that (3) implies (4) let $a > 0$ and Q a compact subset of S . Since $\psi(t, x) = \psi_t(x)$ is a continuous function on $[0, a+\epsilon) \times S$ into S , then $\psi([0, a] \times Q) = \bigcup_{t \leq a} \psi_t(Q)$ is compact in S . Also (3) clearly implies that $\psi_t(x) \rightarrow x$ as $t \rightarrow 0$.

Hence (4) holds. To see that (5) follows, let us consider the semigroup $\{T_t : t \geq 0\}$ as a semigroup of continuous operators on $C(S)$ with the compact-open topology. If $a > 0$ and $U(Q, \epsilon) = \{f \in C(S) : |f(x)| < \epsilon \text{ for all } x \in Q\}$ is a neighborhood of 0 in this topology on $C(S)$ then $V(K, \epsilon) = \{f \in C(S) : |f(x)| < \epsilon \text{ for all } x \in K\} = \bigcup_{t \leq a} \psi_t(Q)$ is also a neighborhood of 0, and if $f \in V(K, \epsilon)$ then $T_t f = f \circ \psi_t \in U(Q, \epsilon)$ for all $t \leq a$. Hence $\{T_t : t \leq a\}$ is compact-open equicontinuous on $C(S)$. Furthermore, since $\psi_t(x) \rightarrow x$ as $t \rightarrow 0$ then $T_t f \rightarrow f$ pointwise on S for all $f \in C(S)$. It now follows from [13, p. 204] that $T_t f \rightarrow f$ weakly in $C(S)$ with the compact-open topology. By Lemma 6.4, $T_t f \rightarrow f$ in the compact-open topology for all $f \in C(S)$ and hence $T_t f \rightarrow f$ in $C(S)_\beta$ as $t \rightarrow 0$.

The transition function of $\{T_t\}$ being $\lambda_t(x) = \psi_t^0(x)$ it follows from the fact that $\bigcup_{t \leq a} \psi_t(Q)$ is compact in S for Q compact in S that the mapping $\mu((t, x)) = \psi_t^0(x)$ defined in (2) of Theorem 6.5 satisfies condition E. Since $T_t f \rightarrow f$ in $C(S)_\beta$ as $t \rightarrow 0$ this means the semigroup $\{T_t\}$ satisfies condition (2) of Theorem 6.5 and from this follows (5) of this theorem as a consequence of (1) of Theorem 6.5.

Finally (5) implies that (1) of Theorem 6.5 holds so that by Theorem 6.6, statement (1) holds. This completes the proof.

This concludes our work in this chapter and in this dissertation. We have by no means exhausted the possible applications of our work with kernels to semigroups of operators. The primary intent of this chapter was twofold. First of all, we wished to obtain the kernel representation of semigroups on $C(S)_\beta$ and $C_0(S)$. Secondly, we wanted to show that when the semigroup is given by a transition function of regular measures, then a consideration of the semigroup on $C(S)_\beta$, rather than $(C(S), \|\cdot\|)$ as is the case in Dynkin [11], is both informative and non-restrictive. We believe that Theorems 6.5, 6.6, and Corollary 6.7 establish this.

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BIOGRAPHY

Francis Dennis Sentilles, Jr., was born in Donaldsonville, Louisiana, on August 7, 1941. He attended parochial schools in the area and graduated from Ascension Catholic High School in 1959. The same year he enrolled at Francis T. Nicholls State College in Thibodaux, Louisiana. In October, 1962, he married the former Claire E. Zeringue of Vacherie, Louisiana, and received the Bachelor of Science degree in February, 1963. He attended the University of Arizona on a graduate assistantship in the spring of that year and then began his work at Louisiana State University as a National Aeronautics and Space Administration Fellow in September, 1963, receiving the Master of Science in August, 1965. At this time he is an instructor at Louisiana State University where he is a candidate for the degree of Doctor of Philosophy in the Department of Mathematics.

EXAMINATION AND THESIS REPORT

Candidate: Francis Dennis Sentilles, Jr.

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Approved:

J. R. Dorroh
Major Professor and Chairman

Max Goodrich
Dean of the Graduate School

EXAMINING COMMITTEE:

W. Bynum

H. Cohen

H. S. Collins

G. Pall

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